Geometry of Radon measures via Hölder parameterizations

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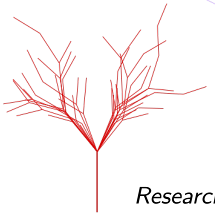

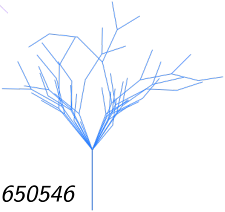

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Decomposition of Measures

Let μ be a measure on a measurable space $(X, \mathcal{M})$.
Let $\mathcal{N} \subseteq \mathcal{M}$ be a family of measurable sets.

- μ is carried by $\mathcal{N}$ if there exist countably many sets $\Gamma_i \in \mathcal{N}$ such that $\mu\left(X \setminus \bigcup_i \Gamma_i\right) = 0$.
- μ is singular to $\mathcal{N}$ if $\mu(\Gamma) = 0$ for every $\Gamma \in \mathcal{N}$.

Exercise (Decomposition Lemma)

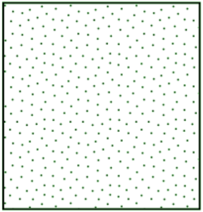
If μ is σ-finite, then μ can be written uniquely as $\mu_{\mathcal{N}} + \mu_{\mathcal{N}}^{\perp}$, where $\mu_{\mathcal{N}}$ is carried by $\mathcal{N}$ and $\mu_{\mathcal{N}}^{\perp}$ is singular to $\mathcal{N}$.

- e.g. $\mathcal{N} = \{A \in \mathcal{M} : \nu(A) = 0\}$: $\mu = \sigma + \rho$ where $\sigma \perp \nu$ and $\rho \ll \nu$
- Proof of the Decomposition Theorem is abstract nonsense.

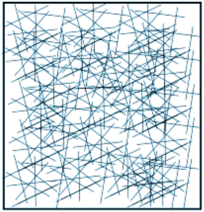
Identification Problem: *Find measure-theoretic and/or geometric characterizations or constructions of $\mu_{\mathcal{N}}$ and $\mu_{\mathcal{N}}^{\perp}$?*

PSA: Don't Think About Support

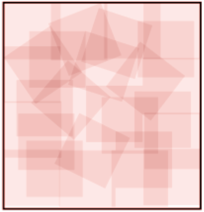
Three Measures. Let $a_i > 0$ be weights with $\sum_{i=1}^{\infty} a_i = 1$.
Let $\{x_i : i \geq 1\}$, $\{\ell_i : i \geq 1\}$, $\{S_i : i \geq 1\}$ be a dense set of points, unit line segments, unit squares in the plane.



$\mu_0 = \sum_{i=1}^{\infty} a_i \, \delta_{x_i}$



$\mu_1 = \sum_{i=1}^{\infty} a_i \, L^1 \llcorner \ell_i$



$\mu_2 = \sum_{i=1}^{\infty} a_i \, L^2 \llcorner S_i$

- μ_0, μ_1, μ_2 are probability measures on $\mathbb{R}^2$
- $\operatorname{spt} \mu$ is smallest closed set carrying μ; $\operatorname{spt} \mu_0 = \operatorname{spt} \mu_1 = \operatorname{spt} \mu_2 = \mathbb{R}^2$
- μ_i is carried by i-dimensional sets (points, lines, squares)
- **The support of a measure is a rough approximation that hides underlying structure of a measure**

Rectifiable Measures: Identification Problem Solved for Absolute Continuous Measures

Let $1 \leq m \leq n-1$ integers. A Radon measure μ on $\mathbb{R}^n$ is ***m*-rectifiable** if μ is carried by images of Lipschitz maps $[0,1]^m \to \mathbb{R}^n$.

μ is **purely *m*-unrectifiable** if μ is singular to Lipschitz images of $[0,1]^m$

Theorem (Azzam, Mattila, Preiss, Tolsa, Toro)

Assume that $\mu \ll \mathcal{H}^m$ $(\Leftrightarrow \overline{\lim}_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} < \infty$ *μ-a.e.) TFAE:*

1. *μ is m-rectifiable*
2. $\underline{\lim}_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} > 0$, $Tan(\mu, x) \subseteq \{c\mathcal{H}^m \llcorner V : V \in G(n,m)\}$ *μ-a.e.*
3. $\lim_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} > 0$ *μ-a.e.*
4. $\underline{\lim}_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} > 0$, $\lim_{r\downarrow 0} \left(\frac{\mu(B(x,r))}{r^m} - \frac{\mu(B(x,2r))}{(2r)^m} \right) = 0$ *μ-a.e.*
5. $\overline{\lim}_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} > 0$, $\int_0^1 \beta_2(\mu, B(x,r))^2 \frac{dr}{r} < \infty$ *μ-a.e., where* $\beta_2(\mu, B(x,r))$ *records "flatness" of μ in* $B(x,r)$

Earlier contributions: Besicovitch, Federer, Marstrand, Morse, Randolph

The study of rectifiability is not done because...

Theorem (Garnett-Killip-Schul 2010)

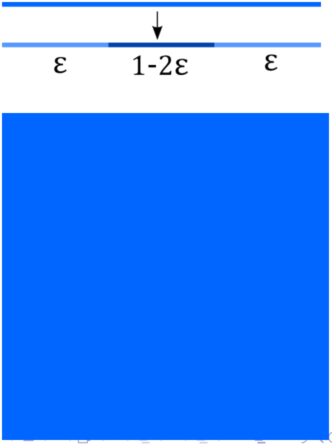
There exist Radon measures μ on $\mathbb{R}^2$ with $\operatorname{spt}\mu = \mathbb{R}^2$ such that μ is 1-rectifiable, $\mu \perp \mathcal{H}^1$, and μ is doubling ($\mu(B(x,2r)) \lesssim \mu(B(x,r))$).

- $\lim_{r\downarrow 0} \dfrac{\mu(B(x,r))}{r} = \infty$ μ-a.e.
- $\displaystyle\int_0^1 \left(\frac{\mu(B(x,r))}{r}\right)^{-1} \frac{dr}{r} < \infty$ μ-a.e.

  (see B-Schul 2016)

- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0,1])$ and $f : [0,1] \to \mathbb{R}^2$ is bi-Lipschitz
- Nevertheless there exist Lipschitz maps $f_i : [0,1] \to \mathbb{R}^2$ such that

$$\mu\left(\mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} f_i([0,1])\right) = 0$$



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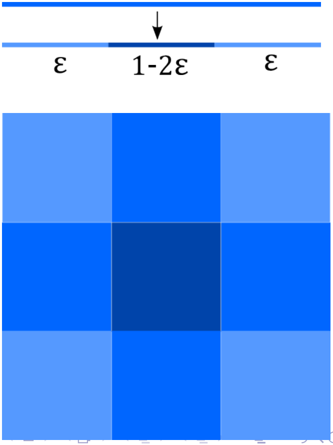

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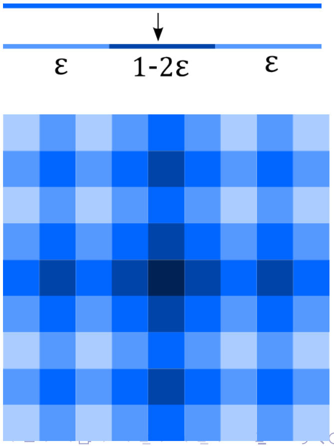

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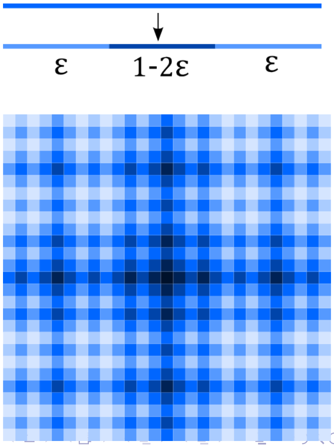

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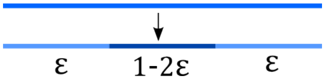

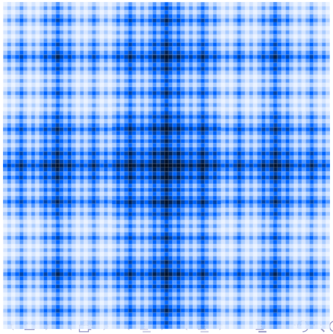

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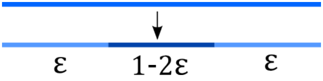
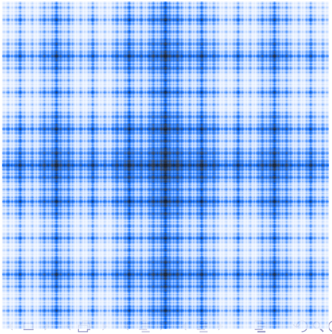




Identification Problem Solved for 1-Rectifiable Measures

Let $1 \leq m \leq n-1$ integers. A Radon measure μ on $\mathbb{R}^n$ is **1-rectifiable** if μ is carried by rectifiable curves (images of Lipschitz maps $[0,1] \to \mathbb{R}^n$)
μ is **purely 1-unrectifiable** if μ is singular to rectifiable curves

Theorem (B, Schul 2017)

Assume that μ is a Radon measure on $\mathbb{R}^n$. TFAE:

1. *μ is 1-rectifiable*
2. $\underline{\lim}_{r\downarrow 0} \frac{\mu(B(x,r))}{r} > 0$ *μ-a.e. and*

$$\sum_{Q\in\Delta} \beta_2^*(\mu, 3000Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) < \infty \quad \mu\text{-a.e.},$$

where $\beta_2^*(\mu, 3000Q)$ records "flatness" of μ in large dilate of a dyadic cube "nonhomogeneously" and "anisotropically"

One new ingredient: L^2 extension of Jones' traveling salesman theorem that works with **non-doubling measures**. Also see Martikainen and Orponen.

What about m-rectifiable measures for $m \geq 2$?

Recent preprints by Azzam-Schul, Edelen-Naber-Valtorta, Ghinassi based on the Reifenberg algorithm give some partial results, but a characterization of 2-rectifiable Radon measures is currently out of reach.

Missing a good characterization of subsets of Lipschitz images of squares. In fact, even the following basic question is wide open.

Open: *Find extra metric, geometric, and/or topological conditions which ensure a compact, connected set $K \subseteq \mathbb{R}^n$ with $\mathcal{H}^2(K) < \infty$ is contained in the image of a Lipschitz map $f : [0,1]^2 \to \mathbb{R}^n$.*

A basic enemy: Let C be the planar four corner Cantor set of dimension 1. Then

$$K = ([0,1]^2 \times \{0\}) \cup (C \times [0,1]) \subset \mathbb{R}^3$$

is connected, compact, and $0 < \mathcal{H}^2(K) < \infty$, but the subset $K' = C \times [0,1]$ is purely 2-unrectifiable.

Current Project (w/ Vellis): Non-integral Dimensions

For each $s \in [1, n]$, let $\mathcal{N}_s$ denote all $(1/s)$-**Hölder curves** in $\mathbb{R}^n$, i.e. all images Γ of $(1/s)$-Hölder continuous maps $f : [0, 1] \to \mathbb{R}^n$.

Decomposition: Every Radon measure μ on $\mathbb{R}^n$ can be uniquely written as $\mu = \mu_{\mathcal{N}_s} + \mu_{\mathcal{N}_s^\perp}$, where

- $\mu_{\mathcal{N}_s}$ is carried by $(1/s)$-Hölder curves
- $\mu_{\mathcal{N}_s^\perp}$ is singular to $(1/s)$-Hölder curves

Notes

- Every measure μ on $\mathbb{R}^n$ is carried by $(1/n)$-Hölder curves (space-filling curves).
- If μ is m-rectifiable, then μ is carried by $(1/m)$-Hölder curves.
- A measure μ is 1-rectifiable iff μ is carried by 1-Hölder curves.
- Martín and Mattila (1988,1993,2000) studied this concept for measures μ of the form $\mu = \mathcal{H}^s \llcorner E$, where $0 < \mathcal{H}^s(E) < \infty$

Essential Examples

"Rectifiable s-sets"

- Let Γ be a generalized von Koch curve of Hausdorff dimension s. Then there exists a $(1/s)$-Hölder map $[0,1] \twoheadrightarrow \Gamma$.
- $\mu = \mathcal{H}^s \llcorner \Gamma$ is carried by $(1/s)$-Hölder curves

"Purely unrectifiable s-sets"

Theorem (Martín and Mattila 1993)

Let $K \subseteq \mathbb{R}^n$ be a self-similar Cantor set of Hausdorff dimension s. Then $\mu = \mathcal{H}^s \llcorner K$ is singular to $(1/s)$-Hölder curves.

- This extends a result of Hutchinson (1981) who showed self-similar Cantor sets of Hausdorff dimension m are purely m-unrectifiable.

Open Problem (Identification Problem for s-sets)
Let $s \in (1,n)$. Characterize s-sets $E \subseteq \mathbb{R}^n$ such that $\mu = \mathcal{H}^s \llcorner E$ is carried by $(1/s)$-Hölder curves. (This is even open when $s = 2$.)

New Results: Measures with Extreme Lower Densities

Theorem (B-Vellis, arXiv 2017)

Let μ *be a Radon measure on* $\mathbb{R}^n$ *and let* $s \in [1, n)$. *Then the measure*

$$\underline{\mu}^s_0 := \mu \llcorner \left\{ x \in \mathbb{R}^n : \underline{\lim}_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} = 0 \right\}$$

is singular to $(1/s)$*-Hölder curves, i.e.* $\underline{\mu}^s_0(\Gamma) = 0$ *for all* $(1/s)$*-Hölder curves* Γ*.*
The measure

$$\underline{\mu}^s_\infty := \mu \llcorner \left\{ x \in \mathbb{R}^n : \int_0^1 \left(\frac{\mu(B(x,r))}{r^s} \right)^{-1} \frac{dr}{r} < \infty \text{ and } \overline{\lim}_{r \downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \right\}$$

is carried by $(1/s)$*-Hölder curves, i.e.* $\underline{\mu}^s_\infty(\mathbb{R}^n \setminus \bigcup_{i=1}^\infty \Gamma_i) = 0$ *for some sequence of* $(1/s)$*-Hölder curves* Γ_i*.*

- At each x, $\int_0^1 \left(\frac{\mu(B(x,r))}{r^s} \right)^{-1} \frac{dr}{r} < \infty$ implies $\lim_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} = \infty$.
  We might call these **points of “rapidly infinite” density**
- The case $s = 1$ obtained earlier by B-Schul (2015, 2016).

Measures with Positive Lower and Finite Upper Density

Corollary

Let μ be a Radon measure on $\mathbb{R}^n$, let $s \in [1, n)$ and $t < s$. Then

$$\mu_+^t := \mu \llcorner \left\{ x \in \mathbb{R}^n : 0 < \underline{\lim}_{r \downarrow 0} \frac{\mu(B(x,r))}{r^t} \leq \overline{\lim}_{r \downarrow 0} \frac{\mu(B(x,r))}{r^t} < \infty \right\}$$

is carried by $(1/s)$-Hölder curves. *(Proof: $t < s$ implies $\mu_+^t \ll \underline{\mu}_\infty^s$)*

Two Refinements

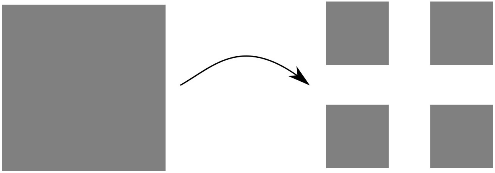
Theorem (B-Vellis, arXiv 2017)

Let μ be a Radon measure on $\mathbb{R}^n$, let $s \in [m, n)$ and $t < s$.
Then μ_+^t is carried by images of (m/s)-Hölder maps $[0,1]^m \to \mathbb{R}^n$.

Theorem (B-Vellis, arXiv 2017)

Let μ be a Radon measure on $\mathbb{R}^n$ and let $t < 1$.
Then μ_+^t is carried by images of bi-Lipschitz maps $[0,1] \to \mathbb{R}^n$.

Example: 2^n-corner Cantor sets



Let $K_t \subset [0,1]^n$ be the self-similar 2^n-corner Cantor set of Hausdorff dimension $t \in (0,n)$. Let $1 \le m \le n-1$ be integers.

- If $t \in [m,n)$, then $\mathcal{H}^t \llcorner K_t$ is singular to (m/t)-Hölder images of $[0,1]^m$ [Martín and Mattila 1993]
- If $t \in [m,n)$, then $\mathcal{H}^t \llcorner K_t$ is carried by (m/s)-Hölder images of $[0,1]^m$ for all $s > t$ [Martín and Mattila 2000] or [B-Vellis]
- If $t \in (0,1)$, then $\mathcal{H}^t \llcorner K_t$ is carried by bi-Lipschitz curves [B-Vellis]

Hölder Parameterization of Leaves of Summable Trees

A **tree off dyadic cube** $\mathcal{T}$ is a collection of dyadic cubes with maximal element Q_0 such that if $Q \in \mathcal{T}$ and $Q \subsetneq Q_0$, then $Q^{\uparrow} \in \mathcal{T}$.

A **leaf** of $\mathcal{T}$ is a limit of a sequence sampled from an infinite branch of $\mathcal{T}$.

Theorem (B-Vellis arXiv 2017)

Let $\mathcal{T}$ be a tree of dyadic cubes (or similar tree of sets). If $s \geq 1$ and

$$\sum_{Q \in \mathcal{T}} (\operatorname{diam} Q)^s < \infty,$$

then $\mathcal{H}^s(\text{Leaves}(\mathcal{T})) = 0$ and there is a $(1/s)$-Hölder curve Γ such that

$$\text{Leaves}(\mathcal{T}) \subseteq \Gamma.$$

- When $s = 1$ this was proved by B-Schul (2016) using the special fact that every connected, compact set with finite $\mathcal{H}^1$ measure is a rectifiable curve.
- When $s > 1$, have to construct the Hölder parameterizations by hand.

Hölder and Bi-Lipschitz Parameterization of Sets of "Small" Assouad Dimension

For $E \subseteq \mathbb{R}^n$, let $\dim_A(E)$ denote its **Assouad dimension**

Theorem (B-Vellis arXiv 2017)

Let $s \in [m, n)$. If $E \subseteq \mathbb{R}^n$ is a bounded set with $\dim_A(E) < s$, then there is an (m/s)-Hölder map $f : [0,1]^m \to \mathbb{R}^n$ such that $E \subseteq f([0,1]^m)$.

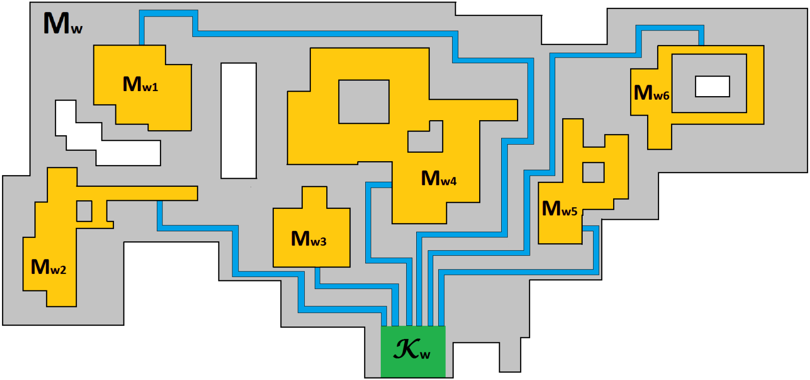
Theorem (B-Vellis arXiv 2017)

If $E \subseteq \mathbb{R}^n$ is a bounded set with $\dim_A(E) < m$ and if the set E is uniformly disconnected in sense of David and Semmes, then there exists a bi-Lipschitz map $f : [0,1]^m \to \mathbb{R}^n$ such that $E \subseteq f([0,1]^m)$.

- When $\dim_A(E) < 1$, the set E is always uniformly disconnected.
- Proof of these results is constructive. Borrows ideas from MacManus' construction of quasicircles passing through uniformly disconnected sets.

Proof of Bi-Lipschitz Parameterization

1. Simple reduction: enough to consider compact sets in the codimension 1 case
2. Use uniform disconnectedness to approximate set by a sequence of manifolds with boundary, ∂M contained in faces of standard grid
3. Construct tree-like surfaces passing through successive approximations:



Takeaways

1. **General Problem in Geometry of Measures:**
   Let $(X, \mathcal{M})$ be a measure space and let $\mathcal{N}$ be a family of measurable sets. Find geometric and/or measure-theoretic characterizations of measures that are
   - carried by $\mathcal{N}$ (rectifiable measures), or
   - singular to $\mathcal{N}$ (purely unrectifiable measures).

   While this problem has been well-studied in $\mathbb{R}^n$ under certain regularity assumptions (absolutely continuous measures), there are many open questions when we drop regularity (Radon measures) or change the space X or choose different sets $\mathcal{N}$.

2. **Non-integral Rectifiability:**
   One candidate for rectifiability in non-integral dimensions based on Hölder continuous images. Some preliminary results have been obtained, but as above there is still more to do!

Thank you