TWO SUFFICIENT CONDITIONS FOR RECTIFIABLE MEASURES

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Abstract. We identify two sufficient conditions for locally finite Borel measures on $\mathbb{R}^n$ to give full mass to a countable family of Lipschitz images of $\mathbb{R}^m$. The first condition, extending a prior result of Pajot, is a sufficient test in terms of $L^p$ affine approximability for a locally finite Borel measure $\mu$ on $\mathbb{R}^n$ satisfying the global regularity hypothesis

$$\limsup_{r \downarrow 0} \mu(B(x,r))/r^m < \infty$$

at $\mu$-a.e. $x \in \mathbb{R}^n$ to be $m$-rectifiable in the sense above. The second condition is an assumption on the growth rate of the 1-density that ensures a locally finite Borel measure $\mu$ on $\mathbb{R}^n$ with

$$\lim_{r \downarrow 0} \mu(B(x,r))/r = \infty$$

at $\mu$-a.e. $x \in \mathbb{R}^n$ is 1-rectifiable.

1. Introduction

In the treatise [Fed69] on geometric measure theory, Federer supplies the following general notion of rectifiability with respect to a measure. Let $1 \leq m \leq n-1$ be integers. Let $\mu$ be a Borel measure on $\mathbb{R}^n$, i.e. a Borel regular outer measure on $\mathbb{R}^n$. Then $\mathbb{R}^n$ is countably $(\mu,m)$ rectifiable if there exist countably many Lipschitz maps $f_i : [0,1]^m \to \mathbb{R}^n$ such that $\mu$ assigns full measure to the images sets $f_i([0,1]^m)$, i.e.

$$\mu \left( \mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i([0,1]^m) \right) = 0.$$ 

When $m = 1$, each set $\Gamma_i = f_i([0,1])$ is a rectifiable curve. Below we shorten Federer’s terminology, saying that $\mu$ is $m$-rectifiable if $\mathbb{R}^n$ is countably $(\mu,m)$ rectifiable.

Two well studied subclasses of rectifiable measures are Hausdorff measures on rectifiable sets and absolutely continuous rectifiable measures. Given any Borel measure $\mu$ on $\mathbb{R}^n$ and Borel set $E \subseteq \mathbb{R}^n$, define the measure $\mu \llcorner E$ (“$\mu$ restricted to $E$”) by the rule $\mu \llcorner E(F) = \mu(E \cap F)$ for all Borel sets $F \subseteq \mathbb{R}^n$. We call a Borel set $E \subseteq \mathbb{R}^n$ an $m$-rectifiable set if $\mathcal{H}^m \llcorner E$ is an $m$-rectifiable measure, where $\mathcal{H}^m$ denotes the $m$-dimensional Hausdorff measure on $\mathbb{R}^n$. One may think of an $m$-rectifiable set $E$ as an $m$-rectifiable measure by identifying $E$ with the measure $\mathcal{H}^m \llcorner E$. More generally, we say that an $m$-rectifiable measure $\mu$ on $\mathbb{R}^n$ is absolutely continuous if $\mu \ll \mathcal{H}^m$, i.e. $\mu(E) = 0$ whenever $E \subset \mathbb{R}^n$ and $\mathcal{H}^m(E) = 0$.

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It is a remarkable fact that rectifiable sets and absolutely continuous rectifiable measures can be identified by the asymptotic behavior of the measures on small balls.

**Definition 1.1 (Hausdorff density).** Let $B(x, r)$ denote the closed ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$. For each positive integer $m \geq 1$, let $\omega_m = \mathcal{H}^m(B^m(0, 1))$ denote the volume of the unit ball in $\mathbb{R}^m$. For all locally finite Borel measures $\mu$ on $\mathbb{R}^n$, we define the **lower Hausdorff m-density** $D^m(\mu, \cdot)$ and upper Hausdorff m-density $\overline{D}^m(\mu, \cdot)$ by

$$D^m(\mu, x) := \liminf_{r \to 0} \frac{\mu(B(x, r))}{\omega_m r^m} \in [0, \infty]$$

and

$$\overline{D}^m(\mu, x) := \limsup_{r \to 0} \frac{\mu(B(x, r))}{\omega_m r^m} \in [0, \infty]$$

for all $x \in \mathbb{R}^n$. If $D^m(\mu, x) = \overline{D}^m(\mu, x)$ for some $x \in \mathbb{R}^n$, then we write $D^m(\mu, x)$ for the common value and call $D^m(\mu, x)$ the **Hausdorff m-density** of $\mu$ at $x$.

**Theorem 1.2 ([Mat75]).** Let $1 \leq m \leq n - 1$. Suppose $E \subset \mathbb{R}^n$ is Borel and $\mu = \mathcal{H}^m \mathbb{1}_E$ is locally finite. Then $\mu$ is $m$-rectifiable if and only if the Hausdorff $m$-density of $\mu$ exists and $D^m(\mu, x) = 1$ at $\mu$-a.e. $x \in \mathbb{R}^n$.

**Theorem 1.3 ([Pre87]).** Let $1 \leq m \leq n - 1$. If $\mu$ is a locally finite Borel measure on $\mathbb{R}^n$, then $\mu$ is $m$-rectifiable and $\mu \ll \mathcal{H}^m$ if and only if the Hausdorff $m$-density of $\mu$ exists and $0 < D^m(\mu, x) < \infty$ at $\mu$-a.e. $x \in \mathbb{R}^n$.

**Remark 1.4.** For any locally finite Borel measure $\mu$ on $\mathbb{R}^n$:

$$\mu \ll \mathcal{H}^m \iff \overline{D}^m(\mu, x) < \infty \text{ at } \mu\text{-a.e. } x \in \mathbb{R}^n; \text{ and,}$$

$$\mu \text{ is } m\text{-rectifiable} \implies D^m(\mu, x) > 0 \text{ at } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

See [Mat95, Chapter 6] and [BS15, Lemma 2.7].

There are several other characterizations of rectifiable sets and absolutely continuous rectifiable measures (e.g. in terms of projections or tangent measures); see Mattila [Mat95] for a full survey of results through 1993. Further investigations on rectifiable sets and absolutely continuous rectifiable measures include [Paj96, Paj97, Lég99, Ler03, Tol12, CGLT14, TT14, Tol14, ADT15, BL14, Bue14, ADT14, AT15, Tol15].

The first result of this note is an extension of Pajot’s theorem on rectifiable sets to absolutely continuous rectifiable measures. To state these results, we must recall the notion of an $L^p$ beta number from the theory of quantitative rectifiability.

**Definition 1.5 ($L^p$ beta numbers).** Let $1 \leq m \leq n - 1$ and let $1 \leq p < \infty$. For every locally finite Borel measure $\mu$ on $\mathbb{R}^n$ and bounded Borel set $Q \subset \mathbb{R}^n$, define $\beta_p^{(m)}(\mu, Q)$ by

$$\beta_p^{(m)}(\mu, Q) := \inf_\ell \int_Q \left( \frac{\text{dist}(x, \ell)}{\text{diam } Q} \right)^p \frac{d\mu(x)}{\mu(Q)} \in [0, 1],$$

where $\ell$ in the infimum ranges over all $m$-dimensional affine planes in $\mathbb{R}^n$. If $\mu(Q) = 0$, then we interpret (1.2) as $\beta_p^{(m)}(\mu, Q) = 0$. 
Remark 1.6. Beta numbers (of sets) were introduced by Jones \[Jon90\] to characterize subsets of rectifiable curves in the plane and are now often called Jones beta numbers. The \(L^p\) variant in Definition 1.5 originated in the fundamental work of David and Semmes on uniformly rectifiable sets \[DS91, DS93\] with the normalization appearing in (1.3). The normalization of \(\beta_{p}(\mu, Q)\) presented in Definition 1.5 is not new; see e.g. \[Ler03\].

When \(Q = B(x, r)\), some sources (e.g. \[DS91, DS93, Paj97\]) define \(L^p\) beta numbers using the alternate normalization

\[
\tilde{\beta}_{p}(\mu, B(x, r))^p := \inf_{\ell} \int_{B(x, r)} \left( \frac{\dist(x, \ell)}{r} \right)^p d\mu(x) \quad \in [0, \infty),
\]

where \(\ell\) in the infimum again ranges over all \(m\)-dimensional affine planes in \(\mathbb{R}^n\). However, \(\beta_{p}(\mu, B(x, r))\) and \(\tilde{\beta}_{p}(\mu, B(x, r))\) are quantitatively equivalent at locations and scales where \(\mu(B(x, r)) \sim r^m\). We have freely translated beta numbers in theorem statements quoted from other sources to the convention of Definition 1.5, which is better suited for generic locally finite Borel measures.

**Theorem 1.7** (\[Paj97\]). Let \(1 \leq m \leq n - 1\) and let

\[
1 \leq p < \infty \quad \text{if } m = 1 \text{ or } m = 2,
\]

\[
1 \leq p < 2m/(m - 2) \quad \text{if } m \geq 3.
\]

Assume that \(K \subset \mathbb{R}^n\) is compact and \(\mu = \mathcal{H}^m \upharpoonright K\) is a finite measure. If \(D^m(\mu, x) > 0\) at \(\mu\)-a.e. \(x \in \mathbb{R}^n\) and

\[
\int_0^1 \beta_{p}(\mu, B(x, r))^2 \frac{dr}{r} < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n,
\]

then \(\mu\) is \(m\)-rectifiable.

In §2, we note the following extension of Pajot’s theorem. Also, see Theorem 2.1.

**Theorem A.** Let \(1 \leq m \leq n - 1\) and let \(1 \leq p < \infty\) satisfy (1.4). Assume that \(\mu\) is a locally finite Borel measure on \(\mathbb{R}^n\) such that \(\mu \ll \mathcal{H}^m\). If \(D^m(\mu, x) > 0\) at \(\mu\)-a.e. \(x \in \mathbb{R}^n\) and (1.5) holds, then \(\mu\) is \(m\)-rectifiable.

In a forthcoming paper, Tolsa \[Tol15\] proves that (1.5) is a necessary condition for an absolutely continuous measure to be rectifiable. Together with Theorem A and (1.1), this result provides a full characterization of absolutely continuous rectifiable measures in terms of the beta numbers and lower Hausdorff density of a measure.

**Theorem 1.8** (\[Tol15\]). Let \(1 \leq m \leq n - 1\) and let \(1 \leq p \leq 2\). If \(\mu\) is \(m\)-rectifiable and \(\mu \ll \mathcal{H}^m\), then (1.5) holds.

**Corollary 1.9.** Let \(1 \leq m \leq n - 1\) and let \(1 \leq p \leq 2\). If \(\mu\) is a locally finite Borel measure on \(\mathbb{R}^n\) such that \(\mu \ll \mathcal{H}^m\), then the following are equivalent:

- \(\mu\) is \(m\)-rectifiable;
- \(D^m(\mu, x) > 0\) at \(\mu\)-a.e. \(x \in \mathbb{R}^n\) and (1.5) holds.

In a companion paper to \[Tol15\], Azzam and Tolsa \[AT15\] prove that in the case \(p = 2\), Theorem A holds with the hypothesis \(D^m(\mu, x) > 0\) at \(\mu\)-a.e. \(x \in \mathbb{R}^n\) on the lower density replaced by a weaker assumption \(\tilde{D}^m(\mu, x) > 0\) at \(\mu\)-a.e. \(x \in \mathbb{R}^n\) on the upper density.
For general $m$-rectifiable measures that are allowed to be singular with respect to $\mathcal{H}^m$, the following basic problem in geometric measure theory is still open.

**Problem 1.10.** For all $1 \leq m \leq n - 1$, find necessary and sufficient conditions in order for a locally finite Borel measure $\mu$ on $\mathbb{R}^n$ to be $m$-rectifiable. (Do not assume that $\mu \ll \mathcal{H}^m$.)

Partial progress on Problem 1.10 has recently been made in [GKS10, BS15, AM15] in the case $m = 1$. In [GKS10], Garnett, Killip, and Schul exhibit a family $(\nu_\delta)_{0 < \delta \leq \delta_0}$ of self-similar locally finite Borel measures on $\mathbb{R}^n$, which are

- **doubling**: $0 < \nu_\delta(B(x,r)) \leq C \nu_\delta(B(x,r/2)) < \infty$ for all $x \in \mathbb{R}^n$ and $r > 0$;
- **badly linearly approximable**: $\beta_2^{(1)}(\nu_\delta, B(x,r)) \geq c_\delta > 0$ for all $x \in \mathbb{R}^n$ and $r > 0$;
- **singular**: $D^1(\nu_\delta, x) = \infty$ at $\nu_\delta$-a.e. $x \in \mathbb{R}^n$ (hence $\nu_\delta \perp \mathcal{H}^1$); and,
- **1-rectifiable**: $\nu_\delta(\mathbb{R}^n \setminus \bigcup_i \Gamma_i) = 0$ for some countable family of rectifiable curves $\Gamma_i$.

In [BS15], Badger and Schul identify a pointwise necessary condition for an arbitrary locally finite Borel measure $\mu$ on $\mathbb{R}^n$ to be 1-rectifiable.

**Theorem 1.11 (BS15 Theorem A).** Let $n \geq 2$ and let $\Delta$ be a system of closed or half-open dyadic cubes in $\mathbb{R}^n$ of side length at most 1. If $\mu$ is a locally finite Borel measure on $\mathbb{R}^n$ and $\mu$ is 1-rectifiable, then

$$\sum_{Q \in \Delta} \beta_2^{(1)}(\mu, 3Q)^2 \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x) < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n.$$ 

The second result of this note is a sufficient condition for a measure $\mu$ with $D^1(\mu, x) = \infty$ at $\mu$-a.e. $x \in \mathbb{R}^n$ to be 1-rectifiable.

**Theorem B.** Let $n \geq 2$ and let $\Delta$ be a system of half-open dyadic cubes in $\mathbb{R}^n$ of side length at most 1. If $\mu$ is a locally finite Borel measure on $\mathbb{R}^n$ and

$$(1.6) \quad \sum_{Q \in \Delta} \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x) < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n,$$

then $\mu$ is 1-rectifiable, and moreover, there exist a countable family of rectifiable curves $\Gamma_i$ and Borel sets $B_i \subseteq \Gamma_i$ such that $\mathcal{H}^1(B_i) = 0$ for all $i \geq 1$ and $\mu(\mathbb{R}^n \setminus \bigcup_{i=1}^\infty B_i) = 0$.

Together Theorem 1.11 and Theorem B provide a full characterization of 1-rectifiability of measures with “pointwise large beta number” (1.7). Examples of measures that satisfy this beta number condition include the measures $(\nu_\delta)_{0 < \delta \leq \delta_0}$ from [GKS10], or more generally, any doubling measure $\mu$ on $\mathbb{R}^n$ whose support is $\mathbb{R}^n$.

**Corollary 1.12.** Let $n \geq 2$ and let $\Delta$ be a system of half-open dyadic cubes in $\mathbb{R}^n$ of side length at most 1. If $\mu$ is a locally finite Borel measure such that

$$(1.7) \quad \liminf_{Q \in \Delta, x \in Q} \frac{\beta_2^{(1)}(\mu, 3Q)}{\text{diam } Q} > 0 \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n,$$

then $\mu$ is 1-rectifiable if and only if (1.6) holds.

Finally, we note that in recent work Azzam and Mourgoglou [AM15] give a weaker condition for 1-rectifiability of a doubling measure with connected support.
Theorem 1.13 ([AM15]). Let $\mu$ be a doubling measure whose support is a topologically connected metric space $X$ and let $E \subseteq X$ be compact. Then $\mu \ll E$ is 1-rectifiable if and only if $D^1(\mu, x) > 0$ for $\mu$-a.e. $x \in E$.

When applied to a doubling measure $\mu$ on $\mathbb{R}^n$ whose support is $\mathbb{R}^n$, Corollary 1.12 and Theorem 1.13 imply that if $D^1(\mu, x) > 0$ at $\mu$-a.e. $x \in \mathbb{R}^n$, then (1.6) holds.

The remainder of this note is split into two sections. We prove Theorem A in §2 and we prove Theorem B in §3.

2. Proof of Theorem A

We show how to reduce Theorem A to Theorem 1.7 using standard geometric measure theory techniques; see Chapters 1, 2, 4, and 6 of [Mat95] for general background. In fact, we will establish the following “localized version” of Theorem A.

Theorem 2.1. Let $1 \leq m \leq n - 1$ and let

$$(2.1) \quad \left\{ \begin{array}{ll}
1 \leq p < \infty & \text{if } m = 1 \text{ or } m = 2, \\
1 \leq p < 2m/(m-2) & \text{if } m \geq 3.
\end{array} \right.$$ 

If $\mu$ is a locally finite Borel measure on $\mathbb{R}^n$ such that

$$J_p(\mu, x) := \int_0^1 \beta_p^{(m)}(\mu, B(x, r))^2 \frac{dr}{r} < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n,$$

then $\mu \ll \{ x \in \mathbb{R}^n : 0 < D^m(\mu, x) \leq D^m(\mu, x) < \infty \}$ is m-rectifiable.

Proof. Without loss of generality, we assume for the duration of the proof that $\mathcal{H}^m$ is normalized so that $\omega_m = \mathcal{H}^m(B^m(0,1)) = 2^m$. This is the convention used in [Mat95].

Suppose that $1 \leq m \leq n - 1$, let $p$ belong to the range (2.1), and let $\mu$ be a locally finite Borel measure on $\mathbb{R}^n$ such that $J_p(\mu, x) < \infty$ at $\mu$-a.e. $x \in \mathbb{R}^n$. Define

$$A := \{ x \in \mathbb{R}^n : 0 < D^m(\mu, x) \leq D^m(\mu, x) < \infty \}.$$ 

Also, for each pair of integers $j, k \geq 1$, define

$$A(j, k) := \{ x \in B(0, 2^k) : 2^{-j}r^m \leq \mu(B(x, r)) \leq 2^jr^m \text{ for all } 0 < r \leq 2^{-k} \}.$$ 

Then $A(j, k)$ is compact and $A(j, k) \subseteq A(j + 1, k + 1)$ for all $j, k \geq 1$. Also note that

$$A = \bigcup_{j, k=1}^\infty A(j, k) = \bigcup_{j, k=1}^\infty A(j, k),$$

Thus, to prove that $\mu \ll A$ is m-rectifiable, it suffices to verify that $\mu \ll A(j, k)$ is m-rectifiable for all $j, k \geq 1$.

Fix any $j, k \geq 1$ and set $K := A(j, k)$, $\nu := \mu \ll K$, and $\sigma := \mathcal{H}^m \ll K$. In order to prove that $\nu$ is m-rectifiable, it is enough to show that $\nu \ll \sigma \ll \nu$ and $\sigma$ is m-rectifiable. By Theorem 6.9 in [Mat95], since $2^{-j-1-m} \leq D^m(\mu, x) \leq 2^{j+1-m}$ for all $x \in K$, we have

$$(2.2) \quad \nu(B(x, r)) = \mu(K \cap B(x, r)) \leq 2^j \mathcal{H}^m(K \cap B(x, r)) = 2^j \sigma(B(x, r))$$

and

$$(2.3) \quad \sigma(B(x, r)) = \mathcal{H}^m(K \cap B(x, r)) \leq 2^{j+1+m} \mu(K \cap B(x, r)) = 2^{j+1+m} \nu(B(x, r))$$
for all $x \in \mathbb{R}^n$ and $r > 0$. Note that

$$\sigma(\mathbb{R}^n) = \sigma(B(0, 2^k)) \leq 2^{j+1+m} \mu(B(0, 2^k)) < \infty,$$

since $\mu$ is locally finite. That is, $\sigma$ is a finite measure. Thus, $\nu$ and $\sigma$ are mutually absolutely continuous by (2.2), (2.3), and Lemma 2.13 in [Mat95]. Now,

$$\sigma(B(x, r)) \leq 2^{j+1+m} \mu(B(x, r)) \leq 2^{2j+2+m/2} \quad \text{for all } x \in K \text{ and } 0 < r \leq 2^{-k-1}.$$  (2.4)

On the other hand, let $K'$ denote the set of $x \in K$ such that

$$2\nu(B(x, r)) = 2\mu(K \cap B(x, r)) \geq \mu(B(x, r)) \quad \text{for all } 0 < r \leq r_x$$

for some $r_x \leq 2^{-k-1}$. Then $\sigma(\mathbb{R}^n \setminus K') = 0$, because $\nu(\mathbb{R}^n \setminus K') = \mu(K \setminus K') = 0$, and

$$\sigma(B(x, r)) \geq 2^{-j-2} \mu(B(x, r)) \geq 2^{-2j-3}r_m \quad \text{for all } x \in K' \text{ and } 0 < r \leq r_x.$$  (2.5)

In particular, $D^m_\sigma(x) < c(m, j) > 0$ at $\sigma$-a.e. $x \in \mathbb{R}^n$. To conclude that $\sigma$ is $m$-rectifiable using Theorem 1.7, it remains to verify $J_p(\sigma, x) < \infty$ at $\sigma$-a.e. $x \in \mathbb{R}^n$.

By (2.4) and (2.5), there exists a constant $C = C(m, j) < \infty$ such that

$$C^{-1} \leq \frac{\nu(B(x, r))}{\sigma(B(x, r))} \leq C \quad \text{for all } 0 < r \leq r_x \text{ at } \sigma\text{-a.e. } x \in \mathbb{R}^n.$$  

Thus, by differentiation of Radon measures, we can write $d\nu = f\,d\sigma$, where $f \in L^1_{\text{loc}}(d\sigma)$ and $C^{-1} \leq f(x) \leq C$ at $\sigma$-a.e. $x \in \mathbb{R}^n$. Therefore, at $\sigma$-a.e. $x \in \mathbb{R}^n$, for every $0 < r \leq r_x$ and for every $m$-dimensional affine plane $\ell$,

$$\int_{B(x, r)} \left( \frac{\text{dist}(y, \ell)}{\text{diam } B(x, r)} \right)^p \frac{d\sigma(y)}{\sigma(B(x, r))} \leq C^2 \int_{B(x, r)} \left( \frac{\text{dist}(y, \ell)}{\text{diam } B(x, r)} \right)^p \frac{d\nu(y)}{\nu(B(x, r))} \leq 2C^2 \int_{B(x, r)} \left( \frac{\text{dist}(y, \ell)}{\text{diam } B(x, r)} \right)^p \frac{d\mu(y)}{\mu(B(x, r))}.$$  

Thus, $\beta^m_p(\sigma, B(x, r))^2 \leq (2C^2)^{2/p} \beta^m_p(\mu, B(x, r))^2$ for all $0 < r \leq r_x$ at $\sigma$-a.e. $x \in \mathbb{R}^n$.

Since $J_p(\mu, x) < \infty$ at $\mu$-a.e. $x \in \mathbb{R}^n$ and $\sigma \ll \mu$, it follows that $J_p(\sigma, x) < \infty$ at $\sigma$-a.e. $x \in \mathbb{R}^n$. Finally, since $K$ is compact, $\sigma = \mathcal{H}^m \ll K$ is finite, and $D^m_\sigma(x) > 0$ and $J_p(\sigma, x) < \infty$ at $\sigma$-a.e. $x \in \mathbb{R}^n$, we conclude that $\sigma$ is $m$-rectifiable by Theorem 1.7.

As noted above, this implies that $\nu = \mu \ll A(j, k)$ is $m$-rectifiable for all $j, k \geq 1$, and therefore, $\mu \ll A$ is $m$-rectifiable. \hfill $\square$

3. Proof of Theorem B

For every Borel measure $\mu$ on $\mathbb{R}^n$, define the quantity

$$S(\mu, x) := \sum_{Q \in \Delta} \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x) \in [0, \infty] \quad \text{for all } x \in \mathbb{R}^n,$$

where $\Delta$ denotes any system of half-open dyadic cubes in $\mathbb{R}^n$ of side length at most 1. Theorem B is a special case of the following statement.

**Theorem 3.1.** Let $n \geq 2$. If $\mu$ is a locally finite Borel measure on $\mathbb{R}^n$, then

$$\rho := \mu \ll \{ x \in \mathbb{R}^n : S(\mu, x) < \infty \}$$

is 1-rectifiable. Moreover, there exists a countable family of rectifiable curves $\Gamma_i \subset \mathbb{R}^n$ and Borel sets $B_i \subseteq \Gamma_i$ such that $\mathcal{H}^1(B_i) = 0$ for all $i \geq 1$ and $\rho(\mathbb{R}^n \setminus \bigcup_{i=1}^\infty B_i) = 0$. 

We start with a lemma, which will be used to organize the proof of Theorem 3.1.

**Lemma 3.2.** Let \( n \geq 1 \) and let \( \mu \) be a locally finite Borel measure on \( \mathbb{R}^n \). Given \( Q_0 \in \Delta \) such that \( \eta := \mu(Q_0) > 0 \) and \( N < \infty \), let

\[
A := \{ x \in Q_0 : S(\mu, x) \leq N \} .
\]

For all \( 0 < \varepsilon < 1/\eta \), the set of dyadic cubes \( Q \subseteq Q_0 \) can be partitioned into good cubes and bad cubes with the following properties:

1. every child of a bad cube is a bad cube;
2. the set \( B := A \setminus \bigcup \{ Q : Q \subseteq Q_0 \text{ is a bad cube} \} \) satisfies \( \mu(B) \geq (1 - \varepsilon \eta)\mu(A) \);
3. \( \sum \text{diam} Q < N/\varepsilon \), where the sum ranges over all good cubes \( Q \subseteq Q_0 \).

**Proof.** Suppose that \( n, \mu, Q_0, \eta, N, \) and \( A \) are given as above and let \( \varepsilon > 0 \). If \( \mu(A) = 0 \), then we may declare every dyadic cube \( Q \subseteq Q_0 \) to be a bad cube and the conclusion of the lemma hold trivially. Thus, suppose that \( \mu(A) > 0 \). Declare that a dyadic cube \( Q \subseteq Q_0 \) is a bad cube if there exists a dyadic cube \( R \subseteq Q_0 \) such that \( Q \subseteq R \) and \( \mu(A \cap R) \leq \varepsilon \mu(A) \mu(R) \). We call a dyadic cube \( Q \subseteq Q_0 \) a good cube if \( Q \) is not a bad cube. Property (1) is immediate. To check property (2), observe that

\[
\mu(A \setminus B) \leq \sum_{\text{maximal bad } Q \subseteq Q_0} \mu(A \cap Q) \leq \varepsilon \mu(A) \sum_{\text{maximal bad } Q \subseteq Q_0} \mu(Q) \leq \varepsilon \mu(A) \mu(Q_0),
\]

where the last inequality follows because the maximal bad cubes are pairwise disjoint (since \( \Delta \) is composed of half-open cubes). Recalling \( \mu(Q_0) = \eta \), it follows that

\[
\mu(B) = \mu(A) - \mu(A \setminus B) \geq (1 - \varepsilon \eta)\mu(A).
\]

Thus, property (2) holds. Finally, since \( S(\mu, x) \leq N \) for all \( x \in A \),

\[
N \mu(A) \geq \int_A S(\mu, x) \, d\mu(x) \geq \sum_{Q \subseteq Q_0} \text{diam} Q \frac{\mu(A \cap Q)}{\mu(Q)} > \varepsilon \mu(A) \sum_{\text{good } Q \subseteq Q_0} \text{diam} Q,
\]

where we interpret \( \mu(A \cap Q)/\mu(Q) = 0 \) if \( \mu(Q) = 0 \). Because \( \mu(A) > 0 \), it follows that

\[
\sum_{\text{good } Q \subseteq Q_0} \text{diam} Q < \frac{N}{\varepsilon}.
\]

This verifies property (3). \( \square \)

**Lemma 3.3.** Let \( n \geq 2 \) and let \( \mu \) be a locally finite Borel measure on \( \mathbb{R}^n \). If

\[
\mu(\{ x \in Q_0 : S(\mu, x) \leq N \}) > 0 \quad \text{for some } Q_0 \in \Delta \text{ and } N < \infty,
\]

then for all \( 0 < \varepsilon < 1/\mu(Q_0) \) the set \( B = B(\mu, Q_0, N, \varepsilon) \) described in Lemma 3.2 lies in a rectifiable curve \( \Gamma \) with \( \mathcal{H}^1(\Gamma) < N/2\varepsilon \) and \( \mathcal{H}^1(B) = 0 \).

**Proof.** Let \( n \geq 2 \) and let \( \mu \) be a locally finite Borel measure on \( \mathbb{R}^n \). Suppose \( \mu(A) > 0 \) for some \( Q_0 \in \Delta \) and \( N < \infty \), where \( A = \{ x \in Q_0 : S(\mu, x) \leq N \} \). Then \( \eta := \mu(Q_0) > 0 \), as well. Given any \( 0 < \varepsilon < 1/\eta \), let \( B = B(\mu, Q_0, N, \varepsilon) \) denote the set from Lemma 3.2. Since \( \varepsilon \) is small enough such that \( \mu(B) \geq (1 - \varepsilon \eta)\mu(A) > 0 \), the cube \( Q_0 \) is a good cube. Construct a connected set \( T \subseteq \mathbb{R}^n \) by drawing a (closed) straight line segment \( \ell_Q \) from
the center of each good cube $Q \subseteq Q_0$ to the center of its parent, which is also a good cube. Let $\overline{T}$ denote the closure of $T$. For all $\delta > 0$,

$$T \subseteq \bigcup_{\text{good } Q \subseteq Q_0} \ell_Q \cup \bigcup_{\text{good } Q \subseteq Q_0} \overline{Q},$$

whence

$$\mathcal{H}_\delta^1(T) \leq \sum_{\text{good } Q \subseteq Q_0} \text{diam } \ell_Q + \sum_{\text{good } Q \subseteq Q_0} \text{diam } \overline{Q} = \sum_{\text{good } Q \subseteq Q_0} \frac{1}{2} \text{diam } Q + \sum_{\text{good } Q \subseteq Q_0} \text{diam } Q.$$

Here we used the fact that any straight line segment $\ell$ can be subdivided into finitely many line segments $\ell'_1, \ldots, \ell'_k$ such that $\text{diam } \ell'_i \leq \delta$ for all $i$ and $\sum_{i=1}^k \text{diam } \ell'_i = \text{diam } \ell$. Since $\sum_{\text{good } Q \subseteq Q_0} \text{diam } Q < N/\varepsilon$, it follows that

$$\mathcal{H}^1(T) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^1(T) \leq \frac{1}{2} \sum_{\text{good } Q \subseteq Q_0} \text{diam } Q < \frac{N}{2\varepsilon}.$$

Now,

$$B \subseteq Q_0 \setminus \bigcup_{\text{good } Q \subseteq Q_0} Q$$

(3.1) $= \bigcup_{i=0}^{\infty} \bigcap_{i=0}^{\infty} Q_i : Q_0 \supseteq Q_1 \supseteq \cdots$ is a chain of good cubes, $\lim_{i \to \infty} \text{diam } Q_i = 0$

(3.2) $\subseteq \left\{ \lim_{i \to \infty} x_i : x_i \in \ell_{Q_i}, \text{ for some good cubes } Q_0 \supseteq Q_1 \supseteq \cdots, \lim_{i \to \infty} \text{diam } Q_i = 0 \right\}.$

Thus, $B \subseteq \overline{T}$ by (3.2). Moreover, refining (3.1), we obtain $B \subseteq \bigcap_{j=1}^{\infty} G_j$, where

$$G_j = \bigcup_{i=j}^{\infty} \bigcap_{i=j}^{\infty} Q'_i : Q'_j \supseteq Q'_{j+1} \supseteq \cdots$$

is a chain of good cubes, $\text{diam } Q'_j \leq 2^{-j}$.

Since $\sum_{\text{good } Q \subseteq Q_0} \text{diam } Q < \infty$, we have $\mathcal{H}_{2^{-j}}^1(G_j) \to 0$, which implies $\mathcal{H}^1(B) = 0$. Finally, because $\overline{T}$ is a continuum in $\mathbb{R}^n$ with $\mathcal{H}^1(T) < \infty$, $T$ coincides with the image $\Gamma = f([0, 1])$ of some Lipschitz map $f : [0, 1] \to \mathbb{R}^n$; e.g. see [DS93, Theorem I.1.8] or [Sch07, Lemma 3.7].

The proof of Theorem 3.1 uses Lemmas 3.2 and 3.3 repeatedly over a suitable, countable choice of parameters.

Proof of Theorem 3.1 Suppose $n \geq 2$ and let $\mu$ be a locally finite Borel measure on $\mathbb{R}^n$. Our goal is to show that $\mu \ll \{x \in \mathbb{R}^n : S(\mu, x) < \infty\}$ is 1-rectifiable. It suffices to prove that $\mu \ll \{x \in Q_0 : S(\mu, x) \leq N\}$ is 1-rectifiable for all $Q_0 \in \Delta$ and for all integers $N \geq 1$.

Fix $Q_0 \in \Delta$ and $N \geq 1$. Let $A = \{x \in Q_0 : S(\mu, x) \leq N\}$. If $\mu(A) = 0$, then there is nothing to prove. Thus, assume $\mu(A) > 0$. Then $\eta = \mu(Q_0) > 0$, as well. Pick any sequence $(\varepsilon_i)_{i=1}^{\infty}$ such that $0 < \varepsilon_i < 1/\eta$ for all $i \geq 1$ and $\varepsilon_i \to 0$ as $i \to \infty$. By Lemmas 3.2
and there exist a Borel set $B_i = B(\mu, Q_0, N, \varepsilon_i) \subseteq A$ and a rectifiable curve $\Gamma_i \supseteq B_i$ such that $H^1(B_i) = 0$ and $\mu(A \setminus B_i) \leq \varepsilon_i \eta \mu(A)$. Hence

$$\mu \left( A \setminus \bigcup_{i=1}^{\infty} \Gamma_i \right) \leq \mu \left( A \setminus \bigcup_{i=1}^{\infty} B_i \right) \leq \inf_{j \geq 1} \mu(A \setminus B_j) \leq \eta \mu(A) \inf \varepsilon_j = 0.$$ 

Therefore, $\mu \ll A$ is 1-rectifiable, and moreover, $\mu \ll A (\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} B_i) = 0$. \hfill \Box

References


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