Singular Points for Two-Phase Free Boundary Problems for Harmonic Measure

Joint work with
Max Engelstein
Tatiana Toro

Matthew Badger

University of Connecticut

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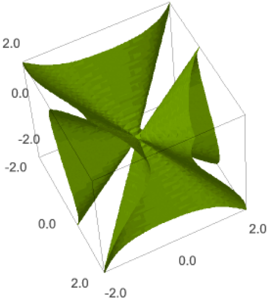

SIAM PDE 2015

New Trends in Elliptic PDE

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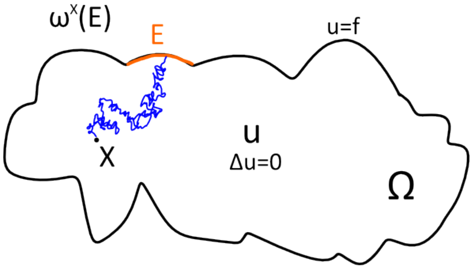
Dirichlet Problem and Harmonic Measure

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain.



Dirichlet Problem

$$(D) \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \end{cases}$$

$$\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2} + \cdots + \partial_{x_n x_n}$$

$\exists!$ family of probability measures $\{\omega^X\}_{X \in \Omega}$ on the boundary $\partial\Omega$ called **harmonic measure** of Ω with pole at $X \in \Omega$ such that

$$u(X) = \int_{\partial\Omega} f(Q) d\omega^X(Q) \qquad \text{solves (D)}$$

By Harnack's inequality, $\omega^X \ll \omega^Y \ll \Omega^X$ for all $X, Y \in \Omega$.

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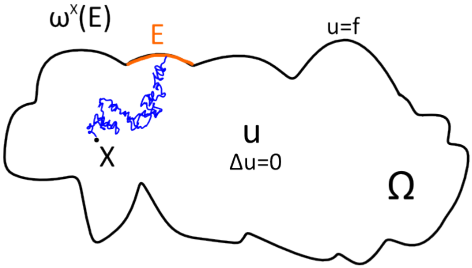

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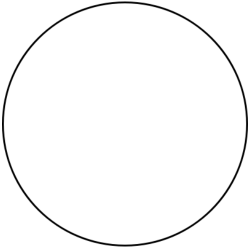
Examples of NTA Domains

NTA domains introduced by Jerison and Kenig 1982:
Quantitative Openness + Quantitative Path Connectedness



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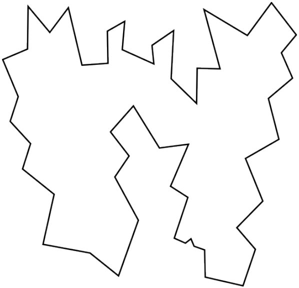

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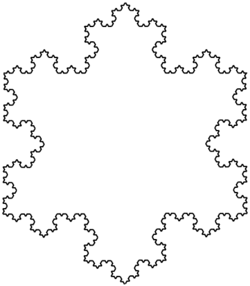

Quasispheres

(e.g. snowflake)

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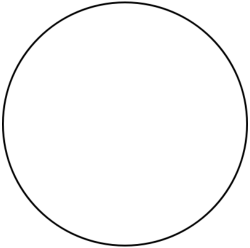

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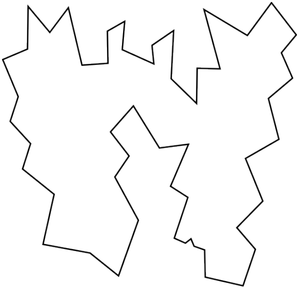

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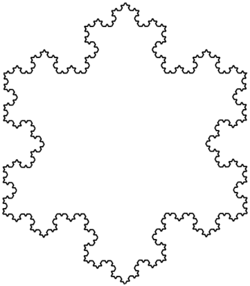

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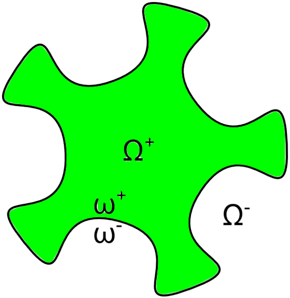
Two-Phase Free Boundary Problem for Harmonic Measure



$\Omega \subset \mathbb{R}^n$ is a **2-sided domain** if:

1. $\Omega^+ = \Omega$ is open and connected
2. $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ is open and connected
3. $\partial\Omega^+ = \partial\Omega^-$

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided domain, equipped with interior harmonic measure ω^+ and exterior harmonic measure ω^-

If the **two-sided kernel** $f = \dfrac{d\omega^-}{d\omega^+}$ *is sufficiently regular,*

then how regular is the boundary $\partial\Omega$?

An Important Example: Polynomial Type Singularities

$\log \frac{d\omega^-}{d\omega^+}$ is smooth $\not\Rightarrow \partial\Omega$ is smooth

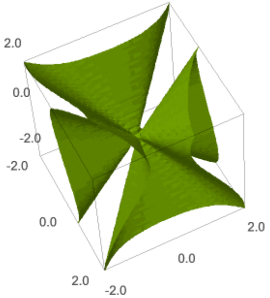


Figure : The zero set of Szulkin's degree 3 harmonic polynomial $p(x, y, z) = x^3 - 3xy^2 + z^3 - 1.5(x^2 + y^2)z$

$\Omega^\pm = \{p^\pm > 0\}$ is a 2-sided domain, $\omega^+ = \omega^-$ (pole at infinity), $\log \frac{d\omega^-}{d\omega^+} \equiv 0$ but $\partial\Omega^\pm = \{p = 0\}$ is not smooth at the origin.

Blowups and Pseudo-blowups under Weak Regularity

Theorem (Kenig and Toro 2006)

Assume that $\Omega^+ = \Omega \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ are NTA domains. If $f = \frac{d\omega^-}{d\omega^+}$ satisfies $\log f \in \mathrm{VMO}(d\omega^+)$, then for all sequences $Q_i \in \partial\Omega$ and $r_i > 0$ with $Q_i \to Q \in \partial\Omega$ and $r_i \to 0$, there exists a harmonic polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree at most $d = d(n, NTA)$ and a subsequence (Q_i', r_i') of (Q_i, r_i) such that

$$\frac{\partial\Omega - Q_i'}{r_i'} \to \Sigma_p = \{x \in \mathbb{R}^n : p(x) = 0\} \text{ as } i \to \infty.$$

Moreover, the zero set Σ_p separates $\mathbb{R}^n$ into complimentary 2-sided NTA domains.

Theorem (B 2011)

$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$: for $Q \in \Gamma_k$, each blowup $\Sigma_p = \lim_i \frac{\partial\Omega - Q}{r_i}$ is the zero set of a homogeneous harmonic polynomial of degree k. Moreover, $\omega^\pm(\partial\Omega \setminus \Gamma_1) = 0$.

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Prior Results: Flat Points

2-sided NTA $+ \log \frac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(d\omega^+) \Longrightarrow \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$

Theorem (B 2013)

- Γ_1 *is relatively open in* $\partial\Omega$
- *All pseudo-blowups* $\Sigma_p = \lim \frac{\partial\Omega - Q_i}{r_i}$ *at* $Q \in \Gamma_1$ *are hyperplanes*
- Γ_1 *has Hausdorff dimension* $n-1$

Theorem (B and Lewis 2015)

- Γ_1 *and* $\partial\Omega$ *have Minkowski dimension* $n-1$ $(\dim_H \leq \dim_M)$
- $\partial\Omega \setminus \Gamma_1 = \Gamma_2 \cup \cdots \cup \Gamma_d$ *has Minkowski dimension* $\leq n-2$

Theorem (Engelstein arXiv:1409.4460)

- *Hölder regularity: If* $\log \frac{d\omega^-}{d\omega^+} \in C^{0,\alpha}$, *then* Γ_1 *is* $C^{1,\alpha}$.
- *Higher regularity: If* $\log \frac{d\omega^-}{d\omega^+} \in C^\infty$, *then* Γ_1 *is* C^∞.

New Results: Singular Points

2-sided NTA + $\log \frac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(d\omega^+) \Longrightarrow \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$

Theorem (B-Engelstein-Toro arXiv:1509.03211)

- *For all $1 \le k \le d$, $U_k := \Gamma_1 \cup \cdots \cup \Gamma_k$ is relatively open in $\partial\Omega$*
- *All pseudo-blowups $\Sigma_p = \lim \frac{\partial\Omega - Q_i}{r_i}$ at $Q \in \Gamma_k$ are zero sets of harmonic polynomials of degree at most k such that $\Omega_p^\pm = \{\pm p > 0\}$ are NTA domains.*
- *$\partial\Omega \setminus \Gamma_1 = \Gamma_2 \cup \cdots \cup \Gamma_d$ has Minkowski dimension $\le n - 3$*
- *"Even degree singular set" $\Gamma_2 \cup \Gamma_4 \cup \ldots$ has Hausdorff dimension $\le n - 4$.*
- *When $n \ge 3$, $\partial\Omega \setminus \Gamma_1$ has Newtonian capacity zero.*

Theorem (B-Engelstein-Toro, in preparation)

Assume that $\log \frac{d\omega^-}{d\omega^+} \in C^{0,\alpha}$. At every boundary point $Q \in \partial\Omega$, there is a unique blowup $\Sigma_p = \lim_{r\to 0} \frac{\partial\Omega - Q}{r}$.

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Remarks / Ingredients in the Proof

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Remarks

- Dimension estimates are sharp by example: $\Omega_p^{\pm} = \{\pm p > 0\}$
  - p is Szulkin's polynomial in $\mathbb{R}^3$, $\Gamma_3 = \{0\}$
  - $p(x_1, y_1, x_2, y_2) = x_1^2 - y_1^2 + x_2^2 - y_2^2$ in $\mathbb{R}^4$, $\Gamma_2 = \{0\}$.
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Local Set Approximation: Approximation Numbers

$\mathcal{H}_{n,d} = \{$zero sets $\Sigma_p = \{p = 0\}$ of nonconstant harmonic polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree $\le d$, $0 \in \Sigma_p\}$

For any $A \subseteq \mathbb{R}^n$ nonempty set, $x \in \mathbb{R}^n$, $r > 0$, we define the **bilateral approximation number** $\Theta_A^{\mathcal{H}_{n,d}}(x, r) \in [0, 1]$, which measures how well A resembles some $\Sigma_p \in \mathcal{H}_{n,d}$ in $B(x, r)$:

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All blowups of A at x belong to $\mathcal{H}_{n,d}$ ("x is a $\mathcal{H}_{n,d}$ **point** of A") if and only if $\lim_{r\to 0} \Theta_A^{\mathcal{H}_{n,d}}(x, r) = 0$ [see B-Lewis 2015]

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Key Ingredient: an "Excess Improvement" Type Lemma

$\mathcal{H}_{n,d} = \{$*zero sets* $\Sigma_p = \{p = 0\}$ *of nonconstant harmonic polynomial* $p : \mathbb{R}^n \to \mathbb{R}$ *of degree* $\leq d$, $0 \in \Sigma_p\}$

Theorem (B-Engelstein-Toro arXiv:1509.03211)

Let $n \geq 2$ *and let* $1 \leq k < d$ *(so that* $\mathcal{H}_{n,k} \subsetneq \mathcal{H}_{n,d}$*). There exists a constant* $\delta = \delta(n,k,d) > 0$ *such that for any harmonic polynomial* $p : \mathbb{R}^n \to \mathbb{R}$ *of degree* d *and, for any* $x \in \Sigma_p$,

$$D^\alpha p(x) = 0 \text{ for all } |\alpha| \leq k \quad \Leftrightarrow \Theta^{\mathcal{H}_{n,k}}_{\Sigma_p}(x,r) \geq \delta \text{ for all } r > 0,$$

$$D^\alpha p(x) \neq 0 \text{ for some } |\alpha| \leq k \Leftrightarrow \Theta^{\mathcal{H}_{n,k}}_{\Sigma_p}(x,r) < \delta \text{ for some } r > 0.$$

Moreover, there exists a constant $C = C(n,k,d) > 1$ *such that*

$$\Theta^{\mathcal{H}_{n,k}}_{\Sigma_p}(x,r) < \delta \text{ for some } r > 0 \quad \Rightarrow \quad \Theta^{\mathcal{H}_{n,k}}_{\Sigma_p}(x,sr) < C\, s^{1/k} \text{ for all } s \in (0,1). \qquad (\star)$$

The special case $k = 1$ first appeared in B 2013.

$\mathcal{H}_{n,k}$ points are detectable in $\mathcal{H}_{n,d}$

> $\mathcal{H}_{n,d} = \{$*zero sets* $\Sigma_p = \{p = 0\}$ *of nonconstant harmonic polynomial* $p : \mathbb{R}^n \to \mathbb{R}$ *of degree* $\leq d$, $0 \in \Sigma_p\}$

For all $n \geq 2$ and $1 \leq k \leq d$, there are $\delta > 0$ and $C > 1$ such that if $\Sigma_p \in \mathcal{H}_{n,d}$, $x \in \Sigma_p$ and $r > 0$, then

$$\Theta_{\Sigma_p}^{\mathcal{H}_{n,k}}(x, r) < \delta \;\Rightarrow\; \Theta_{\Sigma_p}^{\mathcal{H}_{n,k}}(x, sr) < C\, s^{1/k}\ \forall\, s \in (0, 1). \qquad (\star)$$

Remarks

- $\mathcal{H}_{n,k}$ points can be detected in zero sets $\Sigma_p \in \mathcal{H}_{n,d}$ by finding a good enough approximation at a single, coarse scale.
- At $\mathcal{H}_{n,k}$ points in $\Sigma_p \in \mathcal{H}_{n,d}$, blow-ups $(\Sigma_p - x)/r$ converge as $r \to 0$ at a uniform rate for all sufficiently small r.
- Proof: Lojasiewicz type inequalities for harmonic polynomials.
- In LSA framework, property $(\star)$ is called “detectability”. Gives structure theorems for sets with pseudoblowups in $\mathcal{H}_{n,d}$.

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For all $n \geq 2$ and $1 \leq k \leq d$, there are $\delta > 0$ and $C > 1$ such that if $\Sigma_p \in \mathcal{H}_{n,d}$, $x \in \Sigma_p$ and $r > 0$, then

$$\Theta^{\mathcal{H}_{n,k}}_{\Sigma_p}(x, r) < \delta \;\Rightarrow\; \Theta^{\mathcal{H}_{n,k}}_{\Sigma_p}(x, sr) < C\, s^{1/k} \;\forall\, s \in (0, 1). \qquad (\star)$$

Remarks

- $\mathcal{H}_{n,k}$ points can be detected in zero sets $\Sigma_p \in \mathcal{H}_{n,d}$ by finding a good enough approximation at a single, coarse scale.
- At $\mathcal{H}_{n,k}$ points in $\Sigma_p \in \mathcal{H}_{n,d}$, blow-ups $(\Sigma_p - x)/r$ converge as $r \to 0$ at a uniform rate for all sufficiently small r.
- Proof: Lojasiewicz type inequalities for harmonic polynomials.
- In LSA framework, property $(\star)$ is called "detectability". Gives structure theorems for sets with pseudoblowups in $\mathcal{H}_{n,d}$.

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Sets Which are Locally Well Approximated by $\mathcal{H}_{n,d}$

Theorem (B-Engelstein-Toro arXiv:1509.03211)

Let $A \subseteq \mathbb{R}^n$ be closed. Assume that all pseudo-blowups

$$\lim_{i\to\infty} (A - x_i)/r_i \quad (x_i \to x \in A,\ r_i \to 0)$$

of A belong to $\mathcal{H}_{n,d}$, or equivalently, assume that

$$\lim_{r\to 0} \sup_{x\in K} \Theta_A^{\mathcal{H}_{n,d}}(x,r) = 0 \quad \forall\, K \subset\subset A.$$

Then $A = A_1 \cup A_2 \cup \cdots \cup A_d$ where

- *For all $1 \leq k \leq d$, $U_k := A_1 \cup \cdots \cup A_k$ is relatively open in A*
- *$x \in A_k$ if and only if all blowups $\lim_i \frac{A-x}{r_i}$ of A at x are zero sets of homogeneous harmonic polynomials of degree k.*
- *All pseudo-blowups $\lim_i \frac{A-x_i}{r_i}$ of A at $x \in A_k$ belong to $\mathcal{H}_{n,k}$.*
- *For $k \geq 2$, all* **pseudo-blowups** $\lim_i \frac{A_k - x}{r_i}$ **of** A_k *are contained in "degree k" singular set of harmonic polynomial of degree k.*

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Minkowski Type Volume Estimates

Theorem (Naber and Valtorta arXiv:1403.4176)

For all $\Sigma_p \in \mathcal{H}_{n,d}$,

$$\mathrm{Vol}\left(\{x \in B(0,1/2) : \mathrm{dist}(x,\Sigma_p) \leq r\}\right) \leq (C(n)d)^d\, r.$$

For all $S_p \in \mathcal{SH}_{n,d} := \{S_p = \Sigma_p \cap |Dp|^{-1}(0) : \Sigma_p \in \mathcal{H}_{n,d}, 0 \in S_p\}$

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Using prior work Cheeger, Naber, and Valtorta (2015), we prove:

Theorem (B-Engelstein-Toro arXiv:1509.03211)

Let $\mathcal{H}^*_{n,d} = \{\Sigma_p \in \mathcal{H}_{n,d} : \Omega^\pm_p = \{\pm p > 0\}$ *are NTA domains*$\}$.
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Transfers to estimate $\dim_M \partial\Omega \setminus \Gamma_1 = \Gamma_2 \cup \cdots \cup \Gamma_d \leq n-3$ using Local Set Approximation framework [B-Lewis 2015].

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Open Problems

2-sided NTA + $\log \frac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(d\omega^+) \Longrightarrow \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$

- Are blowups of $\partial\Omega$ unique? This is open for Γ_1 too.
- Is Γ_k closed when $k \geq 2$? (Would imply $\dim_M \Gamma_{2k} \leq n-4$.)

2-sided NTA + $\log \frac{d\omega^-}{d\omega^+} \in C^0 \Longrightarrow \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$

- Does Γ_1 have locally finite $(n-1)$-Hausdorff measure?
- Is Γ_1 countably $(n-1)$-rectifiable?

2-sided NTA + $\log \frac{d\omega^-}{d\omega^+} \in C^{0,\alpha} \Longrightarrow \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$

- For $k \geq 2$, are pseudo-blowups of Γ_k equal to the "degree k" singular set of a degree k harmonic polynomial?
- Can we give $C^{1,\alpha}$ local parameterizations of $\partial\Omega$ at $x \in \partial\Omega \setminus \Gamma_1$ by open subsets of zero sets of harmonic polynomials?