THEOREM 0.1 (The Binomial Theorem). *Suppose* $a, b \in \mathbb{R}$ *and* $n \in \mathbb{N} \cup \{0\}$. *Then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

PROOF. The proof is by induction.

First suppose $n = 1$. Then

$$\begin{aligned}(a+b)^1 &= \sum_{k=0}^{1} \binom{1}{i} a^{1-i} b^i \\ &= \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 = a + b\end{aligned}$$

hence 1 lies in the truth set T of the binomial theorem.

Now we must shows that $\ell \in T$ implies that $\ell + 1 \in T$. So we assume that $\ell \in T$ or, equivalently, that

$$(a+b)^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} a^{\ell-k} b^k.$$

Next we multiply both sides by $(a+b)$ and see whether we get the formula for $n = \ell + 1$.

The left hand side is

$$(a+b)^{\ell+1}.$$

The question is whether the right hand side becomes the right hand side of the previous equality with ℓ replaced by $\ell + 1$.

Here's the computation:

$$(a+b) \sum_{k=0}^{\ell} \binom{\ell}{k} a^{\ell-k} b^k = \sum_{k=0}^{\ell} \binom{\ell}{k} a^{\ell+1-k} b^k + \sum_{k=0}^{\ell} \binom{\ell}{k} a^{\ell-k} b^{k+1}.$$

We split off the first and last terms of this sum and leave the rest. The first is

$$I = \binom{\ell}{0} a^{\ell+1} b^0.$$

The last is

$$IV = \binom{\ell}{\ell} a^0 b^{\ell+1}.$$

Next we look at the other terms in the sum. They are II and III as given by:

$$II = \sum_{k=1}^{\ell} \binom{\ell}{k} a^{\ell+1-k} b^k$$

and

$$III = \sum_{k=0}^{\ell-1} \binom{\ell}{k} a^{\ell-k} b^{k+1}.$$

It remains to show that

$$II + III = \sum_{k=1}^{\ell} \binom{\ell+1}{k} a^{\ell+1-k} b^k.$$

Here we use a trick. The counter (k) is a dummy. Substitute $j = k + 1$ in III. So we get

$$III = \sum_{j=1}^{\ell} \binom{\ell}{j-1} a^{\ell-(j-1)} b^j.$$

Or, using k as a counter instead of j:

$$III = \sum_{k=1}^{\ell} \binom{\ell}{k-1} a^{\ell-(k-1)} b^k.$$

Then:

$$\begin{aligned} II + III &= \sum_{k=1}^{\ell} \binom{\ell}{k} a^{(\ell+1)-k} b^k + \sum_{k=1}^{\ell} \binom{\ell}{k-1} a^{(\ell+1)-k} b^k \\ &= \sum_{k=1}^{\ell} \left[\binom{\ell}{k} + \binom{\ell}{k-1} \right] a^{(\ell+1)-k} b^k \\ &= \sum_{k=1}^{\ell} \binom{\ell+1}{k} a^{(\ell+1)-k} b^k \end{aligned}$$

where the last equality follows from the property of binomial coefficients shown in class — the property that defines Pascal's triangle. □