sets, functions, and notation

We have an intuitive notion of sets, elements and "belongs to." If \( A, B \) and \( C \) are sets, \( x \) is an element of (or belonging to) \( A \), we write \( x \in A \).

**Standard set operations:**

1. \( A \cup B \Leftrightarrow \{ x \mid (x \in A) \lor (x \in B) \} \) \hspace{1cm} \text{union}
2. \( A \cap B \Leftrightarrow \{ x \mid (x \in A) \land (x \in B) \} \) \hspace{1cm} \text{intersection}

(Aside: These are actually definitions, but they are written in the form of logically equivalent statements.)

* Why is it appropriate to do so?

3. \( A \setminus B = \{ x \in A \mid x \notin B \} \) \hspace{1cm} \text{not an element of}

   \hspace{1cm} \text{this is the complement of } B \text{ in } A.

   \hspace{1cm} \setminus \text{ is set-theoretic subtraction.}

4. \( C_B = \{ x \in B' \} \) \hspace{1cm} \text{is the complement of } B. \text{ The problem with this definition is that it does not specify where } x \text{ does belong. We only use this notation when there is an implicit universe } U. \text{ Then it means,}

   \hspace{1cm} : = U \setminus B.

* Recall Venn diagrams & interesting examples of sets.

Here's another example, due to Russell, which shows us that not every collection can be a set.
The Russell paradox:
Let \( M \) be the set of all sets which do not contain themselves as elements.

(Aside - it isn't crazy to think of sets of sets. Think of a stamp collection. Each is a collection of stamps as is the collection of all the albums.)

\( \star \) Is \( M \in M? \)

Functions:
In algebra and calculus, we get used to writing functions in the form

\[ y = f(x) := (\text{say}) \ x^2 + 3 \sin(x^2). \]

But functions don't have to have nice "formula-based" expressions. Derivatives of inverse functions are good examples.

But even from equation (1) we can see what is necessary to define a function.

In equation (1) we used a \( := \). This was first used in the computer language Pascal. It means that the object (here \( f(x) \)) on the left \( := \) is defined by the expression to the right.

Notes: a) We need to know which \( x \)'s we're discussing.
   b) What values should the function assume.
   c) What is the rule that associates a value to a particular value of \( x \).
Here's a formal definition

**Defn.** Given two sets $X$ called the **source** or **domain**, $Y$ called the **target** or **range**, a function $f$ from $X$ to $Y$, denoted $f : X \rightarrow Y$

$x \mapsto y = f(x)$

is a rule which associates to each $x \in X$, exactly one element $y \in Y$.

$f(X) := \{ y \in Y \mid \exists x \in X \text{ so that } y = f(x) \}$ is called the **image of $X$ under** $f$.

**Examples:**
1. Often we get sloppy and write $y = f(x)$ and assume $x$ belongs to the set of $x$'s for which $f(x)$ "makes sense."
   a) If $f(x) = \frac{1}{x-5}$, then $X$ can be $\{ x \in \mathbb{R} \mid x \neq 5 \}$
   b) If $f(x) = \ln x$, $X = \{ x \in \mathbb{R}^+ \}$ needs $e \in \mathbb{R}$ is implicit.

* Give some interesting examples of $X, f, Y, f(X)$, for some functions $f$ and their inverses (see below).

**Defn.** Given two functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$

the composition of $g \circ f$ is written $g \circ f : X \rightarrow Z$

$x \mapsto g(f(x))$
One way to think of a function is as a black box. (black is for unknown contents.) All we know about black boxes is what they do, not how they do it.

Boxes like these actually existed. They were called operational amplifiers and, when string together, were called analog computers.

* use $\cdot, +$ to build a particular quadratic polynomial.

**Def:** The identity function is $\text{id}: X \rightarrow X$

\[ x \mapsto (\text{id}(x)) = x. \]

The identity function is defined on every set.

* What makes a function unique? 
* Is the identity function uniquely defined?

**Def:** A function $f: X \rightarrow Y$ is

a) **one-to-one** (1-1) or injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

b) **onto** or surjective if $\forall y \in Y \exists x \in X$ so that $f(x) = y$.

c) **invertible** or bijective if it is both 1-1 and onto.

* Explain a), b), c) in prose. (see text page 1)
Inverse Functions & Inverse Problems -

Suppose we have a function $f : X \to Y$. If we take a point $y \in Y$ and ask which $x$ do you come from, i.e. find $x$ so that $y = f(x)$, we are posing the inverse problem.

An inverse problem asks "where did you come from" or, in other contexts, "what could have caused you." In medicine this takes the form of diagnosis, given some symptoms, what was the disease that caused them. We'd like to specify the cause, then we would know how to treat it. Back to the mathematics — does $y$ specify $x$? If so, we get another function $g : Y \to X$,

$g : Y \to X$

which is "take y back to its roots."

Now let's get serious. We can only ask where you come from, if $y$ comes from at least one $x$, i.e. $y = f(x)$,

* Show that the previous condition is $Y = f(X)$ or $f$ is surjective.

Now let's continue the medical analogy. We want the symptoms to specify the cause. More precisely, given $y$ there is only one $x$ so that $y = f(x)$,

* Show that the previous condition is "$f$ is injective."
So the inverse problem is solvable at all times precisely when \( f \) is bijective or both 1-1 and onto.

**Defn:** Suppose \( f : X \rightarrow Y \). Then \( g : Y \rightarrow X \) is the inverse function to \( f \) if 

\[ g(y) = x \text{ precisely when } y = f(x). \]

**Using black boxes.**

\[ \text{Diagram of } f \circ g = 1_Y \text{ and } g \circ f = 1_X \]

**Why do these diagrams work?**

- If \( \text{id}_X \) is the identity on the set \( X \in Y \)

\[ \text{id}_Y \]

show that \( g : Y \rightarrow X \) is the inverse function of \( f : X \rightarrow Y \)

iff

\[ g \circ f = \text{id}_X \ \text{ and } \ f \circ g = \text{id}_Y. \]