x is not a prime. So x is divisible by some number \( \neq 1 \), hence it is divisible by a prime. But dividing x by any prime gives a remainder of 1. That is a contradiction.
I think of mathematics as I do woodworking —

sometimes I build things — my furniture or barn —

and sometimes I build tools. The analogy works

for me. Some of mathematics is theorem proving —

the building of tools. Some is result oriented — we

solve problems outside the tool constructions.

Was ist Mathematik —

In German, it means "What is Mathematics" and it

is the title of a fascinating book by Courant & Robbins.

There are good stories about both of them — I'll try to spice

up my lectures with such stories. We don't let the truth

get in the way of a good story.

We start with

1) some undefined notions — we understand them on

some intuitive level, but we can't make logical

sense of them.

2) some statements, called axioms, which we take to

be true in the system we study — in fact, they

define the system.

For example, the United States was founded

axiomatically. The country was started with

the statement, "We take these truths to be

self-evident."

3) a logical system — we've just constructed one — that allows

us to make new true statements which are based on the

axioms & logical reasoning.
Theorems are ways of making new true statements; sometimes we use new definitions.

Examples

1. We can define a circle, in Euclidean geometry, using the notions of points & distance (radius). We can then prove that the ratio of the area of the disk, bounded by the circle, to the square of the radius, is a number which doesn't depend on the radius. We have a new tool.

2. In calculus, we define the derivative of a function. If

\[ f(x) = x^2 + 95x \]

then \[ \frac{df}{dx} = \frac{d}{dx} x^2 + \frac{d}{dx} (95x) \]

That computation uses a theorem - the derivative of a sum is the sum of the derivatives.

So, how we prove results and add to our tool chest?

First of all, theorems aren't the only things we need to prove. Provable statements come in a variety of types.

- Theorems - the really important one.
- Propositions - less important but still worth giving a "stand-alone" status.
- Lemmas - these are helper theorems. In fact, that's what they're called in German — Hilfsersätze.
How do we prove a theorem?

The most standard form of a theorem has already been stated.

It has the form
1) set the stage
2) state the hypothesis
then it asserts that
3) the conclusion is true.

1) includes an axiom system
2) is a statement that we assume is true - otherwise there's nothing that has to be proved
3) must be derivable from 1), 2) & statements that are known to be true whenever 1) & 2) are true.

**Theorem: (Gauss)** If \( n \) is any whole number, then
\[
1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}
\]

Implicit that we know what a whole number is \( \frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{n} \) arithmetic.

**What are proofs?**

1) **direct method.**

If the statement is \( P \Rightarrow Q \). We assume \( P \) is true

\[ P \text{ true } \Rightarrow P \text{ true } \Rightarrow P_2 \text{ true } \Rightarrow \ldots \Rightarrow P_n \text{ true } \Rightarrow Q \text{ true.} \]
Ex: \((a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\)

The proof depends on the axioms of arithmetic (assumed true).
We actually only know how to multiply 2 numbers.

Also distributivity (more on this later),
\[
(a+b)^3 \cdot (a+b) (a+b) (a+b) = (a+b) [(a+b)(a+b)]
\]
\[
= (a+b) [a (a+b) + b (a+b)]
\]
\[
= (a+b) [a^2 + ab + ba + b^2]
\]
\[
= (a+b) [a^2 + 2ab + b^2]
\]

etc.

2) Proof by contradiction. We saw that
\[
(P \Rightarrow Q) \iff \neg (P \land \neg Q)
\]
We assume \(P\) is true & \(Q\) is false
Then \(P \land \neg Q\) is true
If we show \(P \land \neg Q\) is false then
\[
\neg (P \land \neg Q)\] must be true
\[
\neg P \Rightarrow Q\] is true.
Euclid's proof is an example.

Here's another method. Prove the contrapositive.

Theorem: \(\sqrt{2}\) is irrational

This statement assume that there is some number system in which we can write \(\sqrt{2}\). Here's a statement that doesn't need real numbers.

Theorem. There is no rational number \(r\) so that \(r^2 = 2\).
Proof: (by proving the contrapositive)

Assume that such a number \( r \) exists, i.e.,

\[ r = \frac{a}{b} \]

satisfies \( b^2 \equiv 2 \pmod{4} \).

We can cancel any common factors \( \neq 2 \) still get the same result. So we can assume \( p^2 \equiv 2 \pmod{4} \).

So \( p^2 \equiv 2 \pmod{4} \).

And \( p^2 \equiv 2 \pmod{4} \) is even, i.e., \( p^2 \) is divisible by 2.

We write this as \( 2 \| p^2 \).

\[ \star \]

Only even \#s have even squares, \( \Rightarrow 2 | p^2 \).

So \( p^2 \) has 4 as a factor, i.e.,

\[ p^2 = 4p = 2q^2 \]

\[ \Rightarrow 2 | q^2 \Rightarrow 2 | q \] (by \( \star \)).

Now we've shown that \( 2 | p^2 \Rightarrow 2 | q^2 \), which means \( p \neq q \) have the common factor 2.

We've contradicted (\( \star \)). So \( r \) cannot exist.

So no such \# \( r \) can exist.

\[ \Box \]

\[ \star \]

If \( p \) is any prime then there is no rational number \( r \) so that \( r^2 = p \).

Next, we develop number systems. Since we want to study calculus, we need to know what real numbers are and what properties they have.