Let's look at the sentence
"I'm lying."

Can it be true?

1) Well, if I am actually lying, then I'm making a true statement so I'm not lying.
2) The only other possibility is that I'm telling the truth so I'm not lying, so the statement is false.

So true $\Rightarrow$ false (what does $\Rightarrow$ mean?)
and false $\Rightarrow$ true.

This isn't a mathematical statement.

A mathematical statement (for our purposes) is one which is true or false—never both and never neither.

Example: If $T$ is a right triangle with hypotenuse of length $c$ and sides of lengths $a$ & $b$, then $c^2 = a^2 + b^2$.

We use special terms, triangle, right triangle, hypotenuse, & side. Also we use lengths, $=, +$.

These terms need to be defined.

The statement has a grammatical (or logical) structure.
It is "If _, then _" called an implication.

Here's another example.
If \( M \) is a compact 3-manifold and \( \pi_1 M = \{1\} \) then
\( M \) is a topological 3-sphere.

(*) You do the dissection. Grisha Perelman proved it last week was awarded the Field's medal for it. He turned it down!

We'll spend the next few lectures discussing mathematical statements.

First we have to know how to define the objects we study. Mathematical definitions have a more-or-less standard structure. There is some variation to cut down on boredom.

**Defn:** In the Euclidean plane \( \mathbb{R}^2 \),
the **circle** with center \( p \) and radius \( r > 0 \)
denoted \( C_r(p) \)
is the set of all points \( q \in \mathbb{R}^2 \) whose distance from \( p \) is \( r \).

Line 1 sets the stage
2 gives the notion being defined
3 (optional) is the notation we use for the notion
4 is the defining property.
Let's try to define a Euclidean triangle.
in class

* Define a quadrilateral.

be careful - these are quadrilaterals

what makes them different?
this is not a quadrilateral

Note - We can't define everything!!

reason - eventually you'll have to define some notion
in terms of itself, this is because a dictionary
has only a finite # of words.

Here's a real proof -
Make a complete list of terms. Pick one term. Delete it
of every term defined using it. Then we cross out all of the terms in the
list. If so, that term can't be defined using any terms, i.e. is not defined.
Otherwise, redo the process using one of the remaining terms. Continue.

So any axiomatic system must have undefined terms.
For example, in set theory, element, set and belongs to are undefined. But we have an intuitive notion of what
they mean:
a set is a collection of things.
An element is a thing (one of those in the collection)

Construct a collection of definitions that is "recursive" i.e. has a
definition defined in terms of itself.

Using definitions of some non-technical words, we can form
sentences - mathematical statements are those which
either true or false, but not both.

Example - The angle sum of a triangle is 2\pi.

In Euclidean geometry, this sentence is true.
In hyperbolic geometry and in spherical geometry, it is false

So the sentence is either not mathematical or is sloppy -
The geometry should have been specified. We often choose
to be sloppy. If we're talking about one context - geometry,
it gets boring to include every hypothesis in every
sentence - we can, when it is clear, assume that some
conditions have been included in the sentence.

\text{
\textcolor{red}{Z} \text{This symbol means "dangerous bend".}}
\text{Z}

When you see the words "clearly", "initially", "obviously",
or anything similar and it isn't clear, trivial or obvious,
you're probably faced where there might be errors.
That's the place to read very carefully.