Solutions

1. State the definitions of

a) A set of n vectors \( \{v_1, \ldots, v_n\} \) is said to be linearly independent.

b) Given an \( m \times n \) matrix \( A = (C_1, \ldots, C_n) \) and an \( n \) vector \( v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \), the product \( Av \) is?

c) Given a set of \( n \) vectors \( \{v_1, \ldots, v_n\} \), \( \text{Span}\{v_1, \ldots, v_n\} \) is?

d) A correspondence \( T : R^n \to R^m \) is said to be linear.

Solutions

a) The homogeneous equation \( x_1 v_1 + \cdots + x_n v_n = 0 \) has only the trivial solution \( x_1 = \cdots x_n = 0 \).

b) \( Av = v_1 C_1 + \cdots + v_n C_n. \)

c) \( \text{Span}\{v_1, \ldots, v_n\} = \text{the set of all linear combinations of } \{v_1, \ldots, v_n\}. \)

d) \( T : R^n \to R^m \) is said to be linear if \( T \) satisfies (1) \( T(v + w) = T(v) + T(w) \) and \( T(\alpha v) = \alpha T(v) \) for all vectors \( v, w \) in \( R^n \) and all scalars \( \alpha \).

2. Using row operations, solve the system of linear equations.

\[
\begin{align*}
    x + 3y + z &= 0 \\
    -4x - 9y + 2z &= 1 \\
    -3y - 6z &= -1
\end{align*}
\]

Solution

\[
\begin{pmatrix}
    1 & 3 & 1 & 0 \\
    -4 & -9 & 2 & 1 \\
    0 & -3 & -6 & -1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & 3 & 1 & 0 \\
    0 & 3 & 6 & 1 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & 3 & 1 & 0 \\
    0 & 3 & 6 & 1 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & 0 & -5 & -1 \\
    0 & 3 & 6 & 1 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

Hence, \( x = 5z - 1, y = -2z + \frac{1}{3} \) and \( z \) is a free variable.
3. Let $T : R^3 \rightarrow R^3$ be a linear transformation given by $T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} x_1 + x_3 \\ 2x_1 + x_3 \\ x_2 - x_3 \end{pmatrix}$.

   a) Find the matrix that expresses $T$ by $T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

   b) Is $T$ one-one? Explain your reasoning.

   c) Is $T$ onto? Explain your reasoning.

   **Solutions**

   a) $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$.

   b) $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix} \rightarrow R_2 - 2R_1 \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \rightarrow R_1 - R_2 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \rightarrow R_2 \leftrightarrow R_3 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence the columns of this matrix are linearly independent. In other words, the equation $AX = 0$ has only the trivial solution. Hence, $ker T = 0$ or $T$ is one-to-one.

   c) The span of the columns of $A$ equals the span of $\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$, which is the entire $R^3$. Thus, $T$ is onto.

4. Let $T$ be a linear transformation from $R^2$ into $R^2$. Suppose that $T(\mathbf{v})$ and $T(\mathbf{w})$ are given in Fig 1 below.

   a) Show the vector $T(\mathbf{v} + 2\mathbf{w})$ in the figure along with your reasoning.

   b) Show the vector $T(\mathbf{v} - \mathbf{w})$ in the figure along with your reasoning.

   **Solution**

   Note that $T(\mathbf{v} + 2\mathbf{w}) = T(\mathbf{v}) + 2T(\mathbf{w})$ and $T(\mathbf{v} - \mathbf{w}) = T(\mathbf{v}) - T(\mathbf{w})$ using the definition of a linear transformation. See the solution to (1), d above. Now one can use the parallelogram definition of the addition and subtraction. See below.
\[ T(v + 2w) = T(v) + 2 T(w) \]

\[ T(v - w) = T(v) - T(w) \]
5. Let $A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ be a 3 by 3 matrix. Define a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(v) = Av$, where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

a) Express the homogenous equation that is satisfied by any element $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in $\ker T$.

b) Solve the equation in (a) and find two vectors that span $\ker T$.

c) Suppose that $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is in the image of $T$. Give a non-homogeneous equation satisfied by $b$.

d) Are the column vectors of $A$ linearly dependent or not? Explain your reasoning.

e) Find two vectors that span the image of $T$.

Solutions

a) $A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

b) $\begin{pmatrix} 2 & 1 & 5 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Hence, one gets $x_1 = -2x_3$, $x_2 = -x_3$ and $x_3$ is a free variable. The vector that span $\ker T$ is $x_3 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$. There was an error in (b).

Find two vectors that span $\ker T$ should read ”find a vector that spans $\ker T$”. I gave everyone who solved the above equation the full mark of 6 points regardless of the rest. Consequently, no one was penalized by the error.

c) $A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

d) Since the above sequence of row operations show that that homogeneous equation in (a) has non-trivial solutions, the column vectors are not linearly independent.

e) The image of $T$ is spanned by the column vectors. The row operations in (b) show that the columns 1 and 2 are linearly independent and the column 3 are given as a linear combination of the columns 1 and 2. So the columns 1 and 2 span the images of $T$.

Good Luck!!