Quiz Solutions **Math 227Q-03** Spring 2008

Name:____________________________ Section:__________

Quiz 1

1.

Using **row operations**, solve the system of linear equations.

$$\begin{aligned} x + 3y + z &= 1 \\ -4x - 9y + 2z &= -1 \\ -3y - 6z &= -3 \end{aligned}$$

Solution

The Augmented matrix of the system is $\begin{pmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{pmatrix}$. Using row operations, reduce it to a row reduced echelon matrix as follows:

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{pmatrix} \overset{R_2+4R_1}{\longrightarrow} \begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{pmatrix} \overset{R_3+R_2}{\longrightarrow} \begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\frac{1}{3}R_2}{\longrightarrow} \begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\overset{R_1-3R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is the row reduced echelon matrix. Hence, we have $\begin{aligned} x - 5z &= -2 \\ y + 2z &= 1 \end{aligned}$.

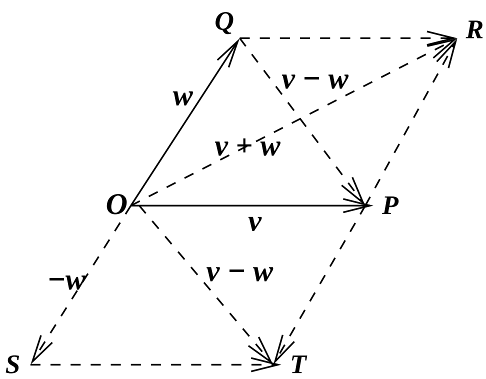
These two equations give us $x = 5z - 2$ and $y = -2z + 1$. This tells us that x and y are given in terms of z, or x and y are dependent variables and z an independent variable. Note that for a given z, say $z = 1$ for instance, $x = 3$ and $y = -1$. This is one set of solutions. There are infinitely many solutions to this problem.

Quiz 2

2. Considering a vector in space as an arrow emanating from the origin O, explain $V \pm W$ geometrically using diagrams, where V, W are two vectors.

Solution

See the figure below, where V, W represent the original vectors given as arrows emanating from the origin. $V + W$ is the diagonal from the origin to O to R in the parallelogram generated by V, W. $V - W$ is the diagonal from O to T of the parallelogram generated by $V, -W$. One may also identify $V - W$ with the diagonal from Q to P.



3.

Quiz 3

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Write **b** as a linear combination of the column vectors of the matrix A.

Solution

Using row operations, solve the vector equation $x_1 \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. The augmented matrix of the equation is $\begin{pmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{pmatrix}$. Now $\begin{pmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{pmatrix} \xrightarrow{R_2 + 3R_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{pmatrix}$ $\xrightarrow{\frac{1}{5}R_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{5} \\ 0 & 5 & 3 & -1 \end{pmatrix} \xrightarrow[R_3 - 5R_2]{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 & -\frac{2}{5} \\ 0 & 1 & 1 & \frac{1}{5} \\ 0 & 0 & -2 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & -1 & -\frac{2}{5} \\ 0 & 1 & 1 & \frac{1}{5} \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[R_2 - R_3]{R_1 + R_3}$ $\begin{pmatrix} 1 & 0 & 0 & \frac{3}{5} \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 1 & 1 \end{pmatrix}$. We see that $x_1 = \frac{3}{5}$, $x_2 = -\frac{4}{5}$, $x_3 = 1$. Thus $\mathbf{b} = \frac{3}{5} \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Quiz 4

4. Given a matrix, $A = \begin{pmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{pmatrix}$, is it possible to find such an h that the third column vector of A is a linear combination of the first two columns? Can you find such h or impossible?

Solution 1

Note that -3 times the first column equals the second column. This implies that the first two columns are linearly dependent, or they are in the same line. Thus, in order for the third column to be in the span of the first two columns, it has to be a multiple of the first (and second) column. This entails that $\begin{pmatrix} 5 \\ -7 \\ h \end{pmatrix} = x \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ for some non-zero number x. But this is impossible because x must satisfies $5 = x, -7 = -3x, h = 2x$ simultaneously. Note the first two equations give $x = 5, x = \frac{3}{7}$, regardless of h. Contradiction! In fact, the third column is independent of the first two columns.

Solution 2

Denote by C_1, C_2, C_3 the columns of the matrix. If C_3 is in the span of C_1, C_2, one should be able to solve the homogeneous equation $x_1C_1 + x_2C_2 + x_3C_3 = 0$ with $x_3 \neq 0$. Now $\begin{pmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{pmatrix} \overset{R_2+3R_1}{\underset{R_3-2R_1}{\longrightarrow}} = \begin{pmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{pmatrix}$. By looking at the second row of this matrix, one sees that only way that this homogeneous equation has a solution implies that $x_3 = 0$, but that means C_3 can never be expressed as a linear combination of C_1, C_2 regardless of h. Thus there is no h that insures that the third column is in that span of the first two columns.

5.

Quiz 5

Let a linear transformation $T : R^3 \to R^2$ be given by the matrix product $T(\mathbf{v}) = A\mathbf{v} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, where $A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

a) Find all vectors in $Ker\,T = \{\mathbf{v} : T(\mathbf{v}) = \mathbf{0}\}$.

b) Verify whether or not T is onto, i.e, the image $T = R^2$.

Solution

a) The elements $\mathbf{v}$ of $Ker\,T$ are solutions to the homogeneous equation $\begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}$. Now $\begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \overset{R_2-2R_1}{\longrightarrow} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \end{pmatrix} \overset{R_1+2R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Thus, $v_1 = -2v_3$, $V_2 = -3v_3$ and v_3 is a free variable. Hence, $\mathbf{v}$ is in $Ker\,T$ if and only if $\mathbf{v} = v_3 \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$ for all choices of v_3.

b) The image of T is the span of the column vectors of A. From the row operations in (a), we see that the first two columns are linearly independent and form the basic vectors for R^2. This implies that any vector in R^2 can be given as a linear combination of the first two columns of A; hence, T is onto.

6.

Quiz 6

Suppose A, B are $n \times n$ matrices. Show AB invertible if A, B are invertible. Show the inverse of AB in terms of the inverses of A, B and justify your answer.

Solution

Suppose AB is not invertible. Then AB regarded as a linear map from R^n into itself is not one-to-one. But this contradicts that A and$B$ are one-to-one regarded as linear maps of R^n because A and B being one-to-one implies AB is one-to-one. Note AB is the composition of two linear maps A and B, i.e., as a linear map $AB = A \circ B$.

7. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. Compute the inverse of A using row operations.

Solution

Solve the matrix equation $AX = I$, where x is the $n \times n$ matrix of unknowns and I is the $n \times n$ identity matrix.

Now $\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \overset{R_1 - 2R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \underset{R_2 - 2R_3}{\overset{R_1 - R_3}{\longrightarrow}} \begin{pmatrix} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$.

Thus the inverse matrix is $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

Quiz 7 Solutions

8. Suppose the determinant of the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find the determinants of the following matrices using the properties of determinants:

(a) $A = \begin{pmatrix} a+2d & b+2e & c+2f \\ d & e & f \\ g & h & i \end{pmatrix}$ (b) $A = \begin{pmatrix} a & b & c \\ d & e & f \\ \alpha g & \alpha h & \alpha i \end{pmatrix}$.

(a) The matrix A is obtained from the original matrix through the row operation $R_1 + 2R_2$. Thus, the determinant is unchanged, i.e., this matrix has 1 as its determinant. (b) This matrix is obtained from the original matrix by the row operation αR_3. Consequently, the determinant of this matrix is $\alpha \times 1 = \alpha$.

9. Let $\mathbf{u} = (1, -2, 0)$, $\mathbf{v} = (0, 1, -2)$, $\mathbf{w} = (0, 0, 3)$. Find the volume of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

The volume of the parallelepiped equals the absolute value of the determinant of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 3 \end{pmatrix}$, which equals 3.

Quiz 8 Solutions

10. Let $\{\mathbf{v_1}, \cdots, \mathbf{v_k}\}$ be k vectors in a vector space V. Show that *Span* $\{\mathbf{v_1}, \cdots, \mathbf{v_k}\}$ is a subspace of V. **One has to show that the span is closed under addition and scalar multiplication.**

Solution

We show that (1) for any two vectors $\mathbf{v}$, $\mathbf{w}$ in the span, $\mathbf{v} + \mathbf{w}$ is in the span and (2) for any scalar α, $\alpha\mathbf{v}$ is in the span. For (1), Let $\mathbf{v} = \sum_{i=1}^{k} a_i \mathbf{v_i}$, $\mathbf{w} = \sum_{i=1}^{k} b_i \mathbf{v_i}$. Then $\mathbf{v} + \mathbf{w} = \sum_{i=1}^{k} (a_i + b_i)\mathbf{v_i}$. Hence $\mathbf{v} + \mathbf{w}$ is in the span. For (2), $\alpha\mathbf{v} = \alpha(\sum_{i=1}^{k} a_i \mathbf{v_i}) = \sum_{i=1}^{k} (\alpha a_i)\mathbf{v_i}$. Hence, $\alpha\mathbf{v}$ is in the span. This proves (1).

11. Let $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & -1 & 2 \end{pmatrix}$ be a 2×4 matrix. Do the following and justify your answers.

Solution

a) Find two vectors that span *Nul A*.

First A is to be reduced to the row reduced echelon matrix by row operations. Now $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & -1 & 2 \end{pmatrix}$ $\xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 1 & \frac{-1}{2} & 1 \end{pmatrix} \xrightarrow{R_1 + R_2} \begin{pmatrix} 2 & 0 & \frac{-1}{2} & 2 \\ 0 & 1 & \frac{-1}{2} & 1 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & \frac{-1}{4} & 1 \\ 0 & 1 & \frac{-1}{2} & 1 \end{pmatrix}$. The vectors in *Nul A* are the solutions to the matrix equation $AX = \mathbf{0}$. Hence, the solutions are given by $\begin{pmatrix} \frac{1}{4}x_3 - x_4 \\ \frac{1}{2}x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, where x_3, x_4 are free variables. Thus, *Nul A* is spanned by $\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$.

b) Find two vectors that span *Col A*.

From the above row operations, one sees that Columns 1 and 2 are the pivot columns. Hence, the first and second column vector of A span $Col\, A$.

Quiz 9 Solution

12. Given a matrix $A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}$, find bases for $Nul\, A$ and $Col\, A$. Show your work.

Solution

$$A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix} \overset{R_1+R_2}{\longrightarrow} \begin{pmatrix} 0 & -2 & -5 & -3 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix} \underset{R_3+4R_1}{\overset{R_2-3R_1}{\longrightarrow}} \begin{pmatrix} 0 & -2 & -5 & -3 \\ 2 & 0 & 6 & 5 \\ -3 & 0 & -18 & -15 \end{pmatrix}$$
$$\overset{\frac{1}{2}R_2}{\longrightarrow} \begin{pmatrix} 0 & -2 & -5 & -3 \\ 1 & 0 & 6 & 5 \\ -3 & 0 & -18 & -15 \end{pmatrix} \overset{R_3+3R_2}{\longrightarrow} \begin{pmatrix} 0 & -2 & -5 & -3 \\ 1 & 0 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{R_1 \leftrightarrow R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & -2 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\overset{\frac{-1}{2}R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus any null vector is given by $\begin{pmatrix} -6x_3 - 5x_4 \\ -\frac{5}{2}x_3 - \frac{3}{2}x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$, where x_3, x_4 are free variables. Thus $Nul\, A$ is spanned by $\begin{pmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$, which are also linearly independent; hence, they form a basis for $Nul\, A$.

We see from the above row operations that Columns 1 and 2 are the pivot columns; hence, $\begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ -6 \\ 8 \end{pmatrix}$ form a basis for $Col\, A$.

Quiz 10 Solutions

13. The matrices A, B below are row equivalent.

$$A = \begin{pmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1) Find $Rank\, A$ and the dimension of $Nul\, A$.

2) Find bases for $Col\, A$ and $Row\, A$.

3) Find a basis for $Nul\,A$.

1) $B = \begin{pmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{\frac{1}{3}R_2}{\longrightarrow} \begin{pmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{R_1+2R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 2 & -5 & 6 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$= C$.

Then C is the row reduced echelon matrix of A. The first two columns are the pivot columns; hence, $Rank\,A$ = the dimension of $Col\,A = 2$. The remaining columns are non-pivot columns. Thus, the dimension of $Nul\,A = 3$.

2) The first two columns of A form a basis for $Col\,A$. The first two rows of B or C form a basis for $Row\,A$.

3) The system of equations $x_1 = -2x_3 + 5x_4 - 6x_5$, $x_2 = -3x_3 + 4x_4 - 4x_5$ defines $Nul\,A$. Thus $\begin{pmatrix} -2x_3+5x_4-6x_5 \\ -3x_3+4x_4-4x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ gives all null vectors. $\begin{pmatrix} -2x_3+5x_4-6x_5 \\ -3x_3+4x_4-4x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 5 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.
$\begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $Nul\,A$. Note that they are linearly independent and span $Nul\,A$.

Quiz 11 Solution

14. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

If A is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution

The characteristic equation of A is $\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = 3\lambda^2 - \lambda^3 = \lambda^2(3-\lambda) = 0$.

Thus, the roots are 0 (double) and 3.

For $\lambda = 3$, $A - 3I = \begin{pmatrix} 1-3 & 1 & 1 \\ 1 & 1-3 & 1 \\ 1 & 1 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$. $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ $\underset{R_2-R_3}{\overset{R_1+2R_3}{\longrightarrow}} \begin{pmatrix} 0 & 3 & -3 \\ 0 & -3 & 3 \\ 1 & 1 & -2 \end{pmatrix} \overset{R_2+R_1}{\longrightarrow} \begin{pmatrix} 0 & 3 & -3 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix} \overset{\frac{1}{3}R_1}{\longrightarrow} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}$. Thus $x_2 = x_3$, $x_1 = x_3$ and x_3 is a free variable; hence, the eigen space is spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

For $\lambda = 0$, $A - 0I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. This matrix is readily reduced to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ by row operations; hence, $x_1 = -x_2 - x_3$ and x_2, x_3 are free. The corresponding eigen vectors are $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Thus $P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that the column vectors of P are linearly independent.

Quiz 12 Solutions

15.

Let W a set of vectors in a vector space V equipped with an inner product.

a) Show that the orthogonal complement $W^{\perp}$ is a subspace of V.

We want to show that $W^{\perp}$ is closed under addition and scalar multiplication. To this end let $\mathbf{u}, \mathbf{v}$ be vectors in $W^{\perp}$, α any scalar and $\mathbf{w}$ be any vector in W. Then, $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} = 0$ and $(\alpha\mathbf{u}) \cdot \mathbf{w} = \alpha(\mathbf{u} \cdot \mathbf{w}) = 0$. So $W^{\perp}$ is closed under addition and scalar multiplication; hence, a subspace.

b) $W^{\perp\perp}$, the orthogonal complement of $W^{\perp}$, contains W.

Let $\mathbf{u}$ be any vector in $W^{\perp}$ and $\mathbf{w}$ be any vector in W. Be definition, $\mathbf{u} \cdot \mathbf{w} = 0$. This tells us that $\mathbf{w}$ belongs to $W^{\perp\perp}$. Hence $W \subset W^{\perp\perp}$.

Good Luck!!