Ambient Isotopic Approximation of Bézier Curves

J.Li T.J.Peters

Department of Mathematics
University of Connecticut, Storrs

ji.li@uconn.edu
tpeters@cse.uconn.edu

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Outline

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- It is an increasing contemporary interest that focuses on preservation of topological characteristics during the piecewise linear (PL) approximation to a Bézier curve.
- The equivalence relation used here for those topological characteristics is ambient isotopy, a stronger notion of equivalence than homeomorphism.

Ambient isotopy

- *Definition:* An ambient isotopy between two subspaces X and Y of $\mathbb{R}^n$, is a continuous function $H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ satisfying:
  (1) $H(\cdot, 0)$ is the identity,
  (2) $H(X, 1) = Y$ and
  (3) $H(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism for $\forall t \in [0,1]$.
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  If H exists, then X and Y are called ambient isotopic.
- *Remark:* In particular, ambient isotopic simple closed curves are of the same knot type [2].

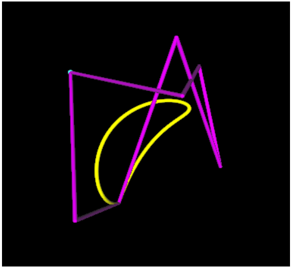
Bézier curves

The parameterized Bézier curve $\mathcal{C}(t)$ in $\mathbb{R}^3$ of degree n is defined to be:

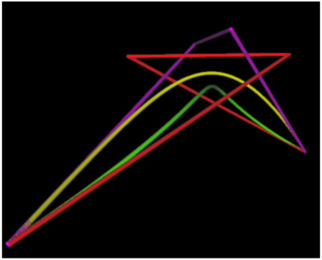
$$\mathcal{C}(t) = \sum_{m=0}^{n} B_{m,n}(t) P_m, t \in [0, 1]$$

where $B_{m,n}(t) = \binom{n}{m} t^m (1-t)^{n-m}$ and each $P_m \in \mathbb{R}^3$ is called a control point. The associated PL curve characterized by the indexed set of the points $\{P_0, P_1, \ldots, P_n\}$ is called a control polygon of the Bézier curve.

Topological differences between Bézier curves and the control polygons



(a) Not homeomorphic



(b) Not ambient isotopic

Figure: Topological differences

De Casteljau's algorithm

- The de Casteljau algorithm uses subdivision to recursively generates new control polygons that more closely approximate the curve under Hausdorff distance [5].

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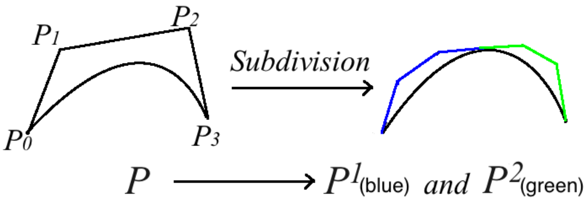
- 

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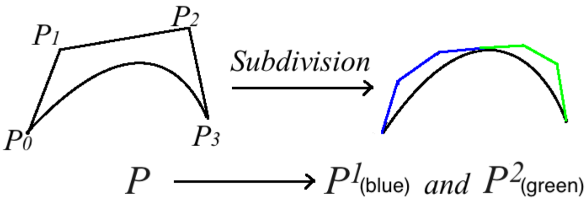
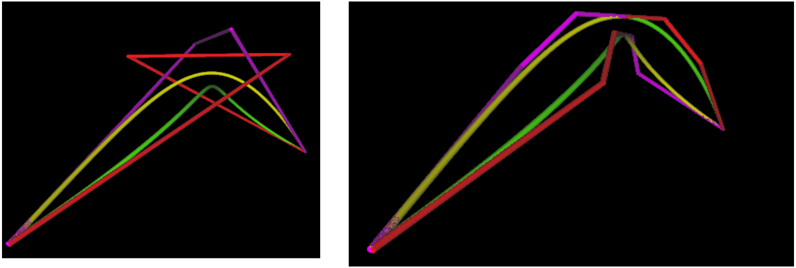
- 

- After i subdivision iterations, there are 2^i sub-control polygons.

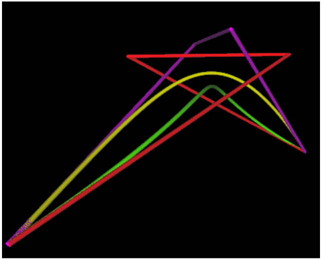
Ambient isotopic Bézier curves and the control polygons

- 

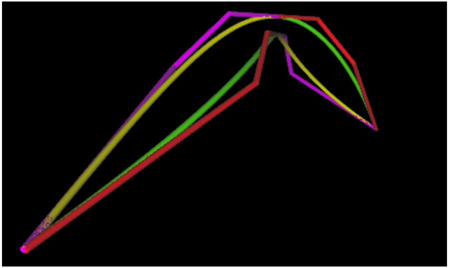
(a) Knotted control polygon (b) Unknotted control polygon

Figure: Ambient isotopy

Ambient isotopic Bézier curves and the control polygons

- 

(a) Knotted control polygon



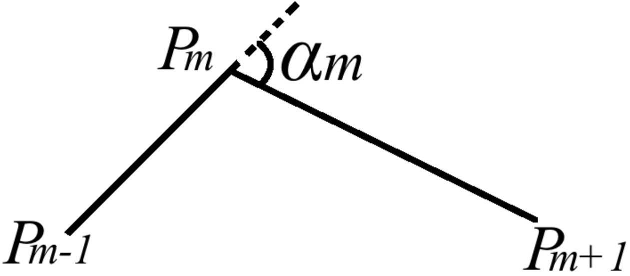
(b) Unknotted control polygon

Figure: Ambient isotopy

- This naturally raises the question of whether this subdivision process will always produce an ambient isotopic PL approximation.

Angular convergence

- **The exterior angle** with measure α_m is formed by $\overrightarrow{P_{m-1}P_m}$ and $\overrightarrow{P_mP_{m+1}}$ and $0 \leq \alpha_m \leq \pi$ for $1 \leq m \leq n-1$.



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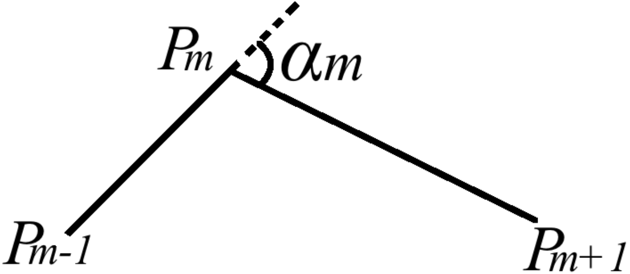


  Intuitively small exterior angles are necessary for good approximations.
- The total curvature of a PL curve is the sum of the exterior angles.

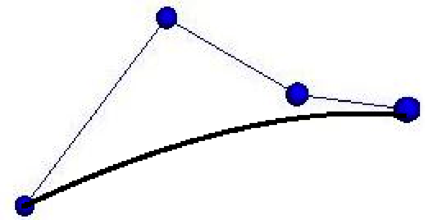
Angular convergence theorem

The symbol $\mathcal{C}$ will denote a simple C^2, regular Bézier curve of arbitrary degree n in $\mathbb{R}^3$.

For $\mathcal{C}$, each exterior angle of the control polygon generated by the i subdivision iterations converges to 0 at a rate of $O(\sqrt{\frac{1}{2^i}})$.

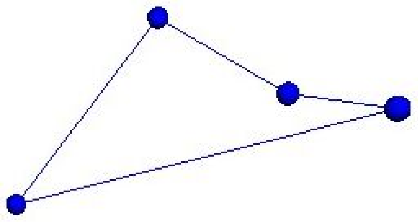
Unknottedness

Consider a sub-control polygon $\mathbf{P}_j$ and the associated sub-curve generated by sufficiently many subdivisions.



Unknottedness

Denote the "closed" PL curve corresponding to $\mathbf{P}_j$ as $\overline{\mathbf{P}}_j$.



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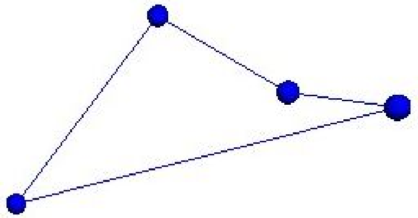

The primary step in the proof is to show that $\overline{\mathbf{P}}_j$ can be deformed into a triangle and the deformation keeps the knot type. Since a triangle is unknotted, the curve $\overline{\mathbf{P}}_j$ needs to be unknotted.

Unknottedness

- After sufficiently many subdivisions, each $\overline{\mathbf{P}}_j$ is unknotted.

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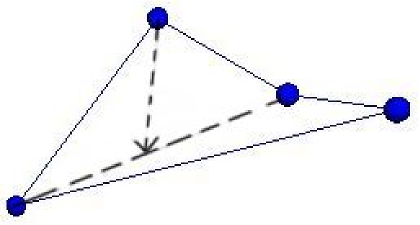
- *Proof:*
  Note that $\kappa(\overline{\mathbf{P}}_j) \leq \kappa(\mathbf{P}_j) + 2\pi$. The number of exterior angles in each $\mathbf{P}_j$ is constant at $n-1$, so $\kappa(\mathbf{P}_j)$ can be made arbitrarily small by the Angular convergence theorem. Thus $\kappa(\overline{\mathbf{P}}_j) < 4\pi$. Then the Fary-Milnor Theorem implies the unknottedness of $\overline{\mathbf{P}}_j$. [4].

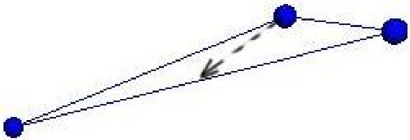
Ambient isotopy

Deforming $\overline{\mathbf{P}}_j$ into a triangle is accomplished by a finite sequence of pushes over the vertices $P_{j,1}, \ldots, P_{j,n-1}$ of $\mathbf{P}_j$.

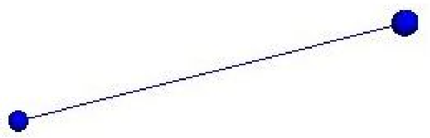
Push



After a push



Another push



Non-intersections

- A *push* is a fundamental function in low-dimensional geometric topology. If a push does not cause intersections, then the push induces an ambient isotopy [1].

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- Two criteria to prevent the intersections:
  1. **local property:** the total curvature of each sub-control polygon is less than $\frac{\pi}{2}$;
  2. **global property:** the control polygon fits inside a non-self-intersecting pipe surface (a tubular neighborhood) [3].

Proof: Angular convergence and convergence in distance.

Ambient isotopic Bézier curves and the control polygons

Theorem (Ambient isotopy)

Sufficiently many subdivisions will yield an ambient isotopic control polygon for a Bézier curve $\mathcal{C}$.

Number of subdivisions to achieve the isotopy

- Furthermore, the control polygon is ambient isotopic to $\mathcal{C}$ via an $O(\sqrt{\frac{1}{2^i}})$ algorithm.

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- Furthermore, the control polygon is ambient isotopic to $\mathcal{C}$ via an $O(\sqrt{\frac{1}{2^i}})$ algorithm.
- Besides, we found a sufficient number of subdivision iterations to achieve ambient isotopic equivalence.

A pipe surface with varying radii

- It is known how to construct a non-self-intersecting pipe surface of "constant" radius [3]. But our work indicates that algorithms would be more efficient if a pipe surface with varying radii is used.

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  Image Credit: http://www.mikeettner.com/2009/05/

Applications

This theory is being applied to visualizing molecular simulations with Bézier curves, under a fundamental requirement to preserve ambient isotopy.

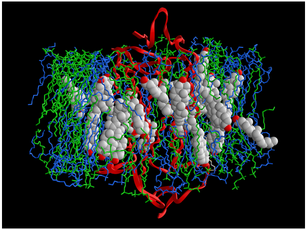

Figure: A molecule in computer visualization

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