Ambient Isotopic Approximation of Bézier Curves

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Abstract

For Bézier curves, subdivision algorithms create piecewise linear (PL) approximations that converge under distance, but it is of increasing contemporary interest to also assess which topological characteristics of the original curve are preserved by these approximations. The equivalence relation used here for those topological characteristics is ambient isotopy, which is stronger than homeomorphism. This paper features two primary results for simple, regular, composite, C^2 Bézier curves:

1. proof of the existence of an ambient isotopic PL approximation under subdivision and
2. determination of a sufficient number of subdivision iterations to achieve ambient isotopic equivalence.

The proofs rely on demonstration that the *exterior angles* of PL curves under subdivision converge to 0 at the rate of $O(\sqrt{\frac{1}{2^i}})$, where i is the number of subdivisions.

Keywords: Bézier curve, knot, subdivision, piecewise linear approximation, angular convergence, non-self-intersection, ambient isotopy.

1 Introduction

A Bézier curve (Definition 2.2) is characterized by an indexed set of points, which form a PL approximation of the curve, called a control polygon (Definition 2.3).

There may be substantial topological differences between Bézier curves and their control polygons. First of all, Bézier curves and their control polygons are not necessarily homeomorphic. There are examples in the literature showing simple (non-self-intersecting) Bézier curves with self-intersecting control polygons or self-intersecting Bézier curves with simple

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control polygons [16, 17]. Secondly, Bézier curves and their control polygons are not necessarily ambient isotopic[1]. An example was constructed recently to show an unknotted Bézier curve with a knotted control polygon [3].

Subdivision algorithms recursively generate PL approximations that more closely approximate a Bézier curve under Hausdorff distance [15, 14]. These PL approximations are actually still control polygons (Definition 2.3) of the curve. Although there may be substantial topological differences between a Bézier curve and its control polygons, we prove that sufficiently many subdivisions will eventually generate control polygons that are ambient isotopic (and therefore, necessarily homeomorphic) to the curve, and we provide the number of subdivision needed. This isotopic equivalence is fundamental within knot theory and has applications in computer graphics, computer animation and scientific visualization.

1.1 Subdivisions

Figure 1 shows the first step of the de Casteljau subdivision on a single Bézier curve. The initial PL approximation P is used as input to generate local PL approximations, P^1 and P^2. Their union, $P^1 \cup P^2$, is then a PL approximation whose Hausdorff distance is closer to the curve than that of P. This illustrative example relies upon generating midpoints of the segments of P, as indicated by providing $\frac{1}{2}$ as an input value to the subdivision algorithm[2].

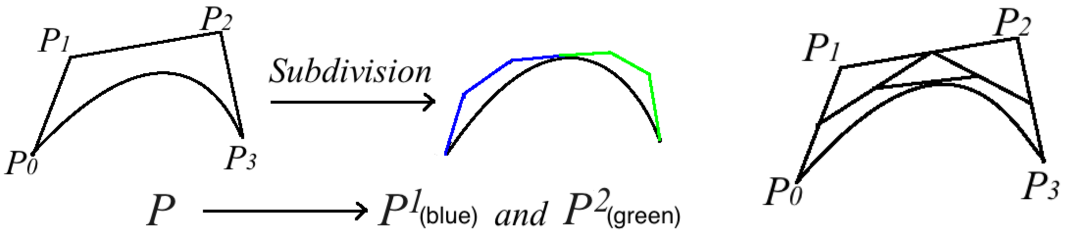

Figure 1: A subdivision with parameter $\frac{1}{2}$

A brief overview is that subdivision proceeds by selecting the midpoint of each segment of P. Then, these midpoints are connected to create new segments. Recursive creation and connection of midpoints continues until a final midpoint is selected. The union of the segments from the last step then forms a PL approximation. Termination is guaranteed since P has only finitely many segments.

1.2 Related work

For subdivision, convergence of the PL curves to a Bézier curve under Hausdorff distance is well known [15], but, to the best of our knowledge, the convergence in terms of angular

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[1] In particular, ambient isotopic simple closed curves are of the same knot type.

[2] Other fractional values can be used, but the analysis given here proceeds by reliance upon $\frac{1}{2}$ and midpoints. The details to change from midpoints are not substantive to the analysis presented here.

measure has not been previously established. The angular convergence is fundamental in proving ambient isotopy between a Bézier curve and its PL approximation.

The result presented here about ambient isotopy for arbitrary degree Bézier curves was discovered when extending the previous theorem of ambient isotopy which restricts to low degree (less than 4) Bézier curves.

For a simple Bézier curve in $\mathbb{R}^3$, we show that a PL approximant under subdivision will eventually also become simple. An existing proof [16] of this relying upon use of the hodograph[3] was devoid of the 3D constructive geometric techniques used here. Furthermore, that previous proof did not provide a specific speed of PL curves becoming simple, but the more direct geometric methods used here easily yield that speed.

2 Definitions and Notation

Exterior angles were defined [11] for closed PL curves, but are adapted here for open curves. Throughout this paper, the standard Euclidean norm will be denoted by $\| \quad \|$.

Definition 2.1 *For an open PL curve with vertices $\{P_0, P_1, \ldots, P_n\}$ in $\mathbb{R}^3$, denote the measures of the exterior angles formed by the oriented line segments to be:*

$$\alpha_1, \ldots, \alpha_{n-1} \textit{ satisfying}$$

$$0 \leq \alpha_m \leq \pi \textit{ for } 1 \leq m \leq n-1.$$

For example, the exterior angle with measure α_m is formed by $\overrightarrow{P_{m-1}P_m}$ and $\overrightarrow{P_mP_{m+1}}$, and $0 \leq \alpha_m \leq \pi$, as that shown in Figure 2. For these open PL curves, it is understood that the exterior angles are not defined at the initial and final vertices.

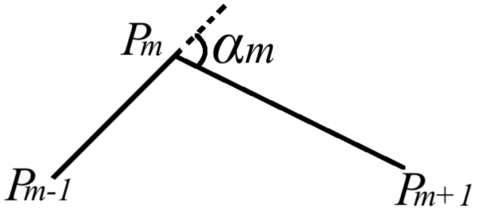

Figure 2: The measure α_m of an exterior angle

The treatment prior to Theorem 4.3 will focus on a single Bézier curve, denoted as $\mathcal{C}$, as this restriction to a single curve greatly simplifies the exposition. From Theorem 4.3 until the start of Section 6, the extensions to the composite case follow directly without any complications, where the composite curve is denoted with bold face, as $\mathbf{C}$. The assumption of C^2 over the entire composite curve is used in the proofs of angular convergence (Theorem 3.1) and ambient isotopic equivalence of the approximation (Theorem 5.2), which are the primary results of this paper. For Section 6, the results are stated for $\mathcal{C}$, but extend naturally over a composite curve, by direct application to each constituent part of the composite curve.

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[3] The derivative of a Bézier curve is also expressed as a Bézier curve, known as the *hodograph* [6].

Definition 2.2 *Denote* $\mathcal{C}(t)$ *as the parameterized Bézier curve of degree* n *with control points* $P_m \in \mathbb{R}^3$*, defined by*

$$\mathcal{C}(t) = \sum_{m=0}^{n} B_{m,n}(t) P_m, t \in [0,1]$$

where $B_{m,n}(t) = \binom{n}{m} t^m (1-t)^{n-m}$.

Definition 2.3 *For* $\mathcal{C}$*, the associated PL curve characterized by the points* $\{P_0, P_1, \ldots, P_n\}$ *is called a* ***control polygon*** *of* $\mathcal{C}$*. After* i *iterations, the subdivision process generates* 2^i *PL sub-curves. Each such PL sub-curve is proved [6] to be a control polygon for the underlying sub-curve, so we call it a* ***sub-control polygon****, denoted by*

$$P^k = (P_0^k, P_1^k, \ldots, P_n^k)$$

for $k = 1, 2, 3, \ldots, 2^i$*. Their union* $(\bigcup_k P^k)$ *forms a new PL curve more closely approximating* $\mathcal{C}$*. The Bézier curve defined by this new PL curve is exactly the Bézier curve* $\mathcal{C}$ *[8], so this PL curve is still a* ***control polygon*** *of* $\mathcal{C}$ *although it is different from the initial one.*

A control polygon can be either an open or closed PL curve, and this will be clear by the context.

Denote the PL curve with vertices $\{P_0, P_1, \cdots, P_n\}$ by $P = (P_0, P_1, \cdots, P_n)$, and the uniform parametrization [13] of P over $[0,1]$ by $l(P)_{[0,1]}$. That is:

$$l(P)_{[0,1]}(\frac{j}{n}) = P_j \ for \ j = 0, 1, \cdots, n$$

and $l(P)_{[0,1]}$ interpolates linearly.

Definition 2.4 *Discrete derivatives [13] are first defined at the parameters* $t_j = \frac{j}{n}$*, where*

$$l(P)_{[0,1]}(t_j) = P_j$$

for $j = 0, 1, \cdots, n-1$*. Let*

$$P'_j = l'(P)_{[0,1]}(t_j) = \frac{P_{j+1} - P_j}{t_{j+1} - t_j}.$$

Denote $P' = (P'_0, P'_1, \cdots, P'_{n-1})$*. Then define the discrete derivative for* $l(P)_{[0,1]}$ *as:*

$$l'(P)_{[0,1]} = l(P')_{[0,1]}.$$

Intuitively, the first discrete derivatives are similar to the tangent lines defined for univariate real-valued functions within a standard introductory calculus course.

In order to avoid many annoying technical considerations and to simplify the exposition, the class of Bézier curves considered will be restricted to those where the derivative never vanishes.

Definition 2.5 *A differentiable curve is said to be regular if its derivative never vanishes.*

Throughout the rest of the presentation, the notation $\mathcal{C}$ will be used for a simple C^2, regular[4] single Bézier curve of arbitrary degree n in $\mathbb{R}^3$. Denote the number of subdivision iterations as i. For convenience, the de Casteljau algorithm is assumed, with a fixed parameter $\frac{1}{2}$.

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[4] The astute reader will note that some of the development does not require that the curve be regular.

3 Angular Convergence

We consider an arbitrary P^k for the following analysis, where, for simplicity of notation, we repress the superscript and denote this arbitrary curve simply as P, where P has the corresponding parameters of the indicated control points by $t_0, t_1, \cdots, t_n$. And let $l(P,i)$ be the uniform parameterization [13] of P on $[\frac{k-1}{2^i}, \frac{k}{2^i}]$ $k \in \{1, 2, 3, \ldots, 2^i\}$. That is

$$l(P,i) = l(P)_{[\frac{k-1}{2^i}, \frac{k}{2^i}]} \quad and \quad l(P,i)(t_m) = P_m \; for \;\; m = \{0, 1, \cdots, n\},$$

where $t_m = \frac{k-1}{2^i} + \frac{m}{n2^i}$. Note from the domain of $l(P,i)$ that

$$t_n - t_0 = \frac{1}{2^i} \;\; and \;\; t_m - t_{m-1} = \frac{1}{n2^i} \;\; for \;\; m = \{1, \cdots, n\}. \tag{1}$$

Furthermore, let

$$\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$$

be the corresponding measures of exterior angles of P (Definition 2.1). Consider the Euclidian distance between the discrete derivatives of the two consecutive segments, that is $||l'(P,i)(t_m) - l'(P,i)(t_{m-1})||$. We will show a rate of $O(\frac{1}{2^i})$ for the convergence

$$||l'(P,i)(t_m) - l'(P,i)(t_{m-1})|| \to 0 \text{ as } i \to \infty.$$

This will imply that $\cos(\alpha_m) \to 1$ with the same rate and that $\alpha_m \to 0$ at a rate of $O(\sqrt{\frac{1}{2^i}})$.

Lemma 3.1 *For $\mathcal{C}$, the value $||l'(P,i)(t_m) - l'(P,i)(t_{m-1})||$ converges uniformly for all m and k to zero at a rate of $O(\frac{1}{2^i})$.*

Proof:

$$||l'(P,i)(t_m) - l'(P,i)(t_{m-1})||$$

$$\leq ||l'(P,i)(t_m) - \mathcal{C}'(t_m)|| + ||\mathcal{C}'(t_m) - \mathcal{C}'(t_{m-1})|| + ||\mathcal{C}'(t_{m-1}) - l'(P,i)(t_{m-1})||. \tag{2}$$

The first and the third terms converge to 0 at a rate of $O(\frac{1}{2^i})$, because the discrete derivative converges to the derivative of the original curve with this rate [13].

Now consider the convergence of the second term. The first derivative $\mathcal{C}'$ satisfies the Lipschitz condition because of it being continuously differentiable on $(0,1)$. So

$$||\mathcal{C}'(t_m) - \mathcal{C}'(t_{m-1})|| \leq \sup_{[0,1]} ||\mathcal{C}''(t)|| \cdot |t_m - t_{m-1}| = \frac{\gamma}{n2^i}, \tag{3}$$

where $\gamma = \sup_{[0,1]} ||\mathcal{C}''(t)||$. And $\sup_{[0,1]} ||\mathcal{C}''(t)||$ is obtained as the maximum of a continuous function on a compact domain because $\mathcal{C}$ is assumed to be C^2. The second equality holds by the Equation (1). Therefore $||l'(P,i)(t_m) - l'(P,i)(t_{m-1})||$ converges to zero at a rate of $O(\frac{1}{2^i})$.

Theorem 3.1 (Angular convergence) *For $\mathcal{C}$, the exterior angles of the PL curves generated by subdivision converge uniformly to 0 at a rate of $O(\sqrt{\frac{1}{2^i}})$.*

Proof: Since $\mathcal{C}(t)$ is assumed to be regular and C^2, the non-zero minimum of $||\mathcal{C}'(t)||$ over the compact set [0,1] is obtained. For brevity, the notations of $u_i = l'(P,i)(t_m)$, $v_i = l'(P,i)(t_{m-1})$ and $\alpha = \alpha_m$ (measure of an arbitrary exterior angle) are introduced. The convergence of u_i to $\mathcal{C}'(t_m)$ [13] implies that $||u_i||$ has a positive lower bound for i sufficiently large, denoted by λ.

Lemma 3.1 gives that $||u_i - v_i|| \to 0$ as $i \to \infty$ at a rate of $O(\frac{1}{2^i})$. This implies: $||u_i|| - ||v_i|| \to 0$ as $i \to \infty$ at a rate of $O(\frac{1}{2^i})$.

Consider the following where the multiplication between vectors is dot product:

$$1 - \cos(\alpha) = 1 - \frac{u_i v_i}{||u_i|| \cdot ||v_i||}$$

$$= \frac{||u_i|| \cdot ||v_i|| - v_i v_i + v_i v_i - u_i v_i}{||u_i|| \cdot ||v_i||}$$

$$\leq \frac{||u_i|| - ||v_i||}{||u_i||} + \frac{||v_i - u_i||}{||u_i||} \leq \frac{||u_i|| - ||v_i||}{\lambda} + \frac{||v_i - u_i||}{\lambda} \tag{4}$$

It follows from Lemma 3.1 that the right hand side converges to 0 at a rate of $O(\frac{1}{2^i})$. Consequently by the above inequality $1 - \cos(\alpha) \to 0$ with the same rate.

It follows from the continuity of $arc\cos$ that α converges to 0 as $i \to \infty$.

Taking the power series expansion of cos we get

$$1 - \cos(\alpha) \geq (\alpha)^2 \cdot (\frac{1}{2} - |\frac{(\alpha)^2}{4!} - \frac{(\alpha)^4}{6!} + \cdots|) = (\alpha)^2 \cdot (\frac{1}{2} - \alpha^2 \cdot |\frac{1}{4!} - \frac{(\alpha)^2}{6!} + \cdots|) \tag{5}$$

Considering the expression on the right hand side of Inequality 5, note that for $1 > \alpha$,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots > |\frac{1}{4!} - \frac{(\alpha)^2}{6!} + \cdots|. \tag{6}$$

Combining Inequality 5 and 6 we have,

$$1 - \cos(\alpha) > (\alpha)^2 \cdot (\frac{1}{2} - \alpha^2 \cdot e).$$

For any $0 < \tau < \frac{1}{2}$, sufficiently many subdivisions will guarantee that α is small enough such that $1 > \alpha$ and $\tau > (\alpha)^2 \cdot e$. Thus

$$1 - \cos(\alpha) > (\alpha)^2 \cdot (\frac{1}{2} - \alpha^2 \cdot e) > (\alpha)^2 \cdot (\frac{1}{2} - \tau) > 0. \tag{7}$$

Then convergence of the left hand side implies that α converges to 0 at a rate of $O(\sqrt{\frac{1}{2^i}})$.

4 Non-self-intersections from Subdivision

Even though the initial PL approximant might not be simple, if the Bézier curve is simple, then subdivision eventually produces a PL approximant that is simple [16], where that proof relied upon use of the hodograph in presenting an existence argument. Consistent with the approach taken here, that result will now be shown by constructive geometric proof, which will lead to further insights, in particular the number of subdivision iterations needed for the PL approximation to also be simple.

Lemma 4.1 is similar to one previously proven [16], where the angles in the previous publication were defined over a different range of values than used here.

The previous definition of exterior angles for open curves (Definition 2.1) was noted as a specialization of the original use for closed curves, where it was created to unify the concept of total curvature for closed curves that were PL or differentiable.

Definition 4.1 *The curvature of $\mathcal{C}$ is given by*

$$\kappa(t) = \frac{||\mathcal{C}'(t) \times \mathcal{C}''(t)||}{||\mathcal{C}'(t)||^3}, \quad t \in [0,1].$$

Its total curvature [4] *is the integral:* $\int_0^1 |\kappa(t)|\, dt$.

Definition 4.2 [11] *For a closed PL curve $\bar{P}$ in $\mathbb{R}^3$, formed from points $P_0, P_1, \ldots, P_n$, its total curvature $\kappa(\bar{P})$ is defined as*

$$\kappa(\bar{P}) = \sum_{m=0}^{n} \alpha_m,$$

where α_0 and α_m are both defined in the interval $[0, \pi]$, where α_0 is formed from the edges $\overrightarrow{P_nP_0}$ and $\overrightarrow{P_0P_1}$, while α_n is formed from the edges $\overrightarrow{P_{n-1}P_n}$ and $\overrightarrow{P_nP_0}$

4.1 Local non-self-intersection

Theorem 4.1, which is know as Fenchel's Theorem [4], is applicable both to PL curves and to differentiable curves.

Theorem 4.1 [11] *The total curvature of any closed curve is at least 2π, with equality holding if and only if the curve is convex.*

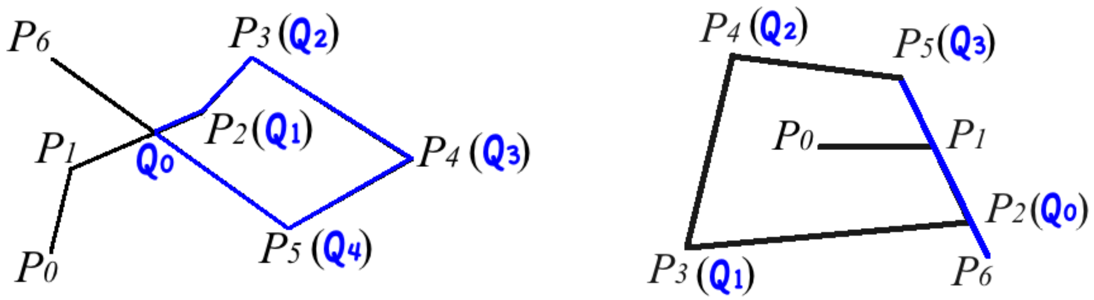
Lemma 4.1 (Non-self-intersection criteria) *Let $P = (P_0, P_1, \cdots, P_n)$ be an open PL curve in $\mathbb{R}^3$. If the total curvature ($\sum_{j=1}^{n-1} \alpha_j$) of P is less than π, then P is simple.*

Proof:

Assume to the contrary that P is self-intersecting. Then there must exist at least one closed loop. (The assumption $\sum_{j=1}^{n-1} \alpha_j < \pi$ precludes the case of two consecutive edges being collinear.) Arbitrarily choose one of such loops. There are two cases to consider: a single point intersection as in Figure 3(a) and the collinear case as in Figure 3(b).

Case 1 (Non-collinear): Label the single intersection as Q_0 to be the first vertex of the loop (Figure 3(a)). Label the other vertices clockwise and denote this loop by

$$\bar{Q} = (Q_0, Q_1, \cdots, Q_{n'}, Q_0),$$



(a) Non-collinear case

(b) Collinear case

Figure 3: Self-intersecting PL curves in 3D

for an appropriately chosen value of n'. Denote the measure of the corresponding exterior angle of $\bar{Q}$ at $Q_0, Q_1, \cdots, Q_{n'}$ by $\beta_0, \beta_1, \cdots, \beta_{n'}$. Note that $\beta_0 \leq \pi$ (Definition 2.1). Note also that each exterior angle β_l (for $l = 1, 2, \cdots, n'$) of $\bar{Q}$ (except the initial angle β_0) is an exterior angle of P. So

$$\sum_{l=0}^{n'} \beta_l = \beta_0 + \sum_{l=1}^{n'} \beta_l \leq \pi + \sum_{j=1}^{n-1} \alpha_j.$$

Since $\sum_{j=1}^{n-1} \alpha_j < \pi$ we get

$$\sum_{l=0}^{n'} \beta_l \leq \pi + \sum_{j=1}^{n-1} \alpha_j < 2\pi.$$

But for the closed loop $\bar{Q}$, $\sum_{l=0}^{n'} \beta_l \geq 2\pi$ (Theorem 4.1), which is a contradiction.

Case 2 (Collinear): The loop is formed by a subset of the vertices $\{P_0, P_1, \cdots, P_n\}$ in this case (Figure 3(b)). Pick an arbitrary point from the subset and relabel it as Q_0. Label the other vertices clockwise and use the notation $\bar{Q} = (Q_0, Q_1, \cdots, Q_{n'}, Q_0)$ and $\beta_0, \beta_1, \cdots, \beta_{n'}$ as the above. Note for this case that, each exterior angle β_l of $\bar{Q}$ is an exterior angle of P. So

$$\sum_{l=0}^{n'} \beta_l \leq \sum_{j=1}^{n-1} \alpha_j.$$

And $\sum_{j=1}^{n-1} \alpha_j < \pi$, so $\sum_{l=0}^{n'} \beta_l < \pi$. But for the closed loop $\bar{Q}$, $\sum_{l=0}^{n'} \beta_l \geq 2\pi$ (Theorem 4.1), which is a contradiction! □

Theorem 4.2 (Simple sub-control polygons) *For $\mathcal{C}$, there exists a sufficiently large value of i, such that after i-many subdivisions, each of the sub-control polygons generated as $P^k = (P_0^k, P_1^k, \ldots, P_n^k)$ for $k = 1, 2, 3, \ldots, 2^i$ will be simple.*

Proof: Since the measures of the exterior angles converge uniformly to zero as i increases (Theorem 3.1) and since each open $P^k = (P_0^k, P_1^k, \ldots, P_n^k)$ has n edges, there exists i sufficiently large such that

$$\sum_{j=1}^{n-1} \alpha_j^k < \pi,$$

for each $k = 1, 2, 3, \dots, 2^i$. Use of Lemma 4.1 completes the proof. □

4.2 Global non-self-intersection

Recall a subdivision takes a control polygon and generates two sub-control polygons. As before, for simplicity, we denote two generated sub-control polygons as $P = (P_0, P_1, \cdots, P_n)$ and $Q = (Q_0, Q_1, \cdots, Q_n)$. Without loss of generality P and Q can be assumed to be ordered and also that P is before Q.

Definition 4.3 *For $\mathcal{C}$, we call two connected sub-control polygons generated by subdivision adjacent. Precisely they satisfy the following properties:*

(1). They are connected in the following sense: the last vertex P_n of the initial sub-control polygon P is the first vertex Q_0 of the subsequent Q, that is, $P_n = Q_0$.

(2). The line segments $\overrightarrow{P_{n-1}P_n}$ and $\overrightarrow{Q_0Q_1}$ are collinear, but do not overlap with each other by the regularity of $\mathcal{C}$. So the exterior angle at the connection point is 0.

Definition 4.4 *Given two points a and b on curve $\mathcal{C}$, let Π_a be the normal plane to the curve at a, and let Π_b be the normal plane to the curve at b. Let Γ_{ab} be the closed region bounded by the planes Π_a and Π_b, and the non-self-intersecting pipe surface $S_c(r)$. We call Γ_{ab} the pipe section determined by a and b.*

Lemma 4.2 *For a simple open regular oriented curve $\mathcal{C}$ in $\mathbb{R}^3$, let $S_c(r)$ be a non-self-intersecting pipe surface for $\mathcal{C}$. Let $\{a_0, a_1, \cdots, a_m\}$ be an indexed set of points on $\mathcal{C}$. Let Γ_j for each $j = 0, 1, \cdots, m-1$ be a pipe section determined by a_j and a_{j+1}* (Definition 4.4). *Then section Γ_j intersects only its two neighbors Γ_{j-1} and Γ_{j+1} at the normal planes Π_j and Π_{j+1}, where Π_j is the normal plane at a_j, for $j = 1, 2, \cdots, m-2$.*

Proof: If two sections intersect in any way other than stated, the non-self-intersecting hypothesis for $S_c(r)$ is violated. □

Lemma 4.3 *For $\mathcal{C}$, consider two adjacent sub-control polygons $P = (P_0, P_1, \cdots, P_n)$ and $Q = (Q_0, Q_1, \cdots, Q_n)$. Let $w = P_n = Q_0$. If the total curvatures of each sub-control polygon is less than $\frac{\pi}{2}$, then the plane Π normal to $\mathcal{C}$ at w separates $P \setminus \{w\}$ and $Q \setminus \{w\}$ into disjoint half-spaces H_1 and H_2.*

Proof: Note that Π separates $\mathbb{R}^3$ into two disjoint open half-spaces, say H_1 and H_2, such that $\mathbb{R}^3 = H_1 \cup \Pi \cup H_2$ and $H_1 \cap H_2 = \emptyset$.

First, by Definition 4.3, two segments $\overrightarrow{P_{n-1}P_n}$ and $\overrightarrow{Q_0Q_1}$ are connected such that $w = P_n = Q_0$, and the exterior angle at the connection point is 0. So $\overrightarrow{P_{n-1}P_n}$ and $\overrightarrow{Q_0Q_1}$ are separated by Π except at the connection point. Then, suppose to the contrary that $P \setminus \{w\}$ and $Q \setminus \{w\}$ are not separated by Π, then one of them must intersect Π (Figure 4 shows an orthogonal projection of a 3D graph).

Assume without loss of generality that a segment $Q_{m-1}Q_m$ intersects Π at the point u for some $m \in \{2, 3, \cdots, n\}$. This would give us a closed PL curve $\bar{R}$ and

$$\bar{R} = (Q_0, Q_1, \cdots, Q_{m-1}, u, Q_0).$$

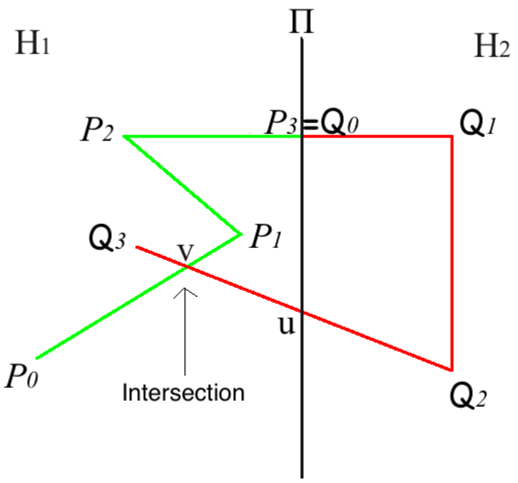


Figure 4: Intersecting adjacent sub-control polygons

Denote the exterior angle of $\bar{R}$ at Q_0 as θ. Since $\overrightarrow{Q_0Q_1}$ is normal to the plane Π and $\overrightarrow{uQ_0} \subset \Pi$, $\overrightarrow{Q_0Q_1} \perp \overrightarrow{uQ_0}$ and hence $\theta = \frac{\pi}{2}$. Denote the exterior angle of $\bar{R}$ at u as γ. Then $\gamma \leq \pi$ (Definition 2.1). Denote the other exterior angles of $\bar{R}$ as β_j. By Theorem 4.1 we have

$$\theta + \sum_{j=1}^{m-1} \beta_j + \gamma \geq 2\pi.$$

Thus

$$\sum_{j=1}^{m-1} \beta_j \geq 2\pi - \theta - \gamma \geq \frac{\pi}{2}.$$

Because all these β_j are also exterior angles of Q, the total curvature of Q is not smaller than $\sum_{j=1}^{m-1} \beta_j$ and hence not smaller than $\frac{\pi}{2}$. But it is a contradiction to the hypothesis on Q with the total curvature less than $\frac{\pi}{2}$. □

Lemma 4.4 *For a Bézier curve $\mathcal{C}$, subdivide the control polygons of $\mathcal{C}$ for i subdivisions such that, the convex hull of the control polygon lies inside a non-self-intersecting pipe surface $S_{\mathcal{C}}(r)$ of $\mathcal{C}$ (Convergence in distance), and each of the resulting sub-control polygons $P^k = (P_0^k, \cdots, P_n^k)$ for $k = 1, \cdots, 2^i$ has a total curvature less than $\frac{\pi}{2}$ (*Theorem 3.1: *Angular convergence). Let Γ_k be the pipe section as in* Definition 4.4 *determined by the points P_0^k and P_n^k for $k = 1, \cdots, 2^i$. Then each sub-control polygon and its associated sub Bézier curve lie inside the corresponding pipe section.*

Proof: Let Π_k be the plane normal to $\mathcal{C}$ at P_0^k for $k = 1, \cdots, 2^i$ and Π_{2^i+1} be the plane normal to $\mathcal{C}$ at $P_n^{2^i}$. Note that for each $k = 1, \cdots, 2^i$, Π_k is normal to the line segment $\overrightarrow{P_0^k P_1^k}$ and Π_{2^i+1} is normal to $\overrightarrow{P_{n-1}^{2^i} P_n^{2^i}}$.

Lemma 4.3 shows that each sub-control polygon P^k does not intersect or cross Π_k or Π_{k+1}, except P_0^k and P_n^k lie on Π_k and Π_{k+1} respectively. Since it lies in $S_{\mathcal{C}}(r)$, it lies in Γ_k.

Since by the convex hull property, the associated sub Bézier curve lies within the same convex region determined by the planes Π_k and Π_{k+1} as P^k and it contained in $S_{\mathcal{C}}(r)$, the associated sub Bézier curve lies in Γ_k as well. □

Lemma 4.4 extends the previously established related result (Lemma 3.2 [12]), that was shown only for cubic Bézier curves and was instrumental in the proof of isotopy under subdivision for low-degree Bézier curves [19]. The result here of isotopy under subdivision for arbitrary degree Bézier curves avoids this previous cubic restriction.

The use of the composite curve bold face notation of $\mathbf{C}$ begins here.

Theorem 4.3 (A simple control polygon) *Sufficient subdivisions will yield a simple control polygon approximating* $\mathbf{C}$.

Proof: Following the paper [10], construct a non-self-intersecting pipe surface $S_c(r)$ of constant radius r about $\mathbf{C}$. Then perform sufficiently many subdivisions such that all sub-control polygons of the curve fit inside the pipe. (Assume the minimal number of subdivisions needed here is ι_1.) Angular convergence (Theorem 3.1) gives that the sum of exterior angles of each sub-control polygon can be arbitrarily small via subdivision. So enough subdivisions will yield a sum less than $\frac{\pi}{2}$. (Assume the minimal number of subdivisions needed here is ι_2.)

The above are sufficient conditions for Lemma 4.1 and Lemma 4.4. Consequently after ι subdivisions, where $\iota = \max\{\iota_1, \iota_2\}$, we get simple sub-control polygons and there is no intersection between sub-control polygons (adjacent or not) except the possible connection points. This implies a simple control polygon for the Bézier curve $\mathbf{C}$. □

5 Ambient isotopy

This section obtains our major result: the existence of an ambient isotopic control polygon (PL approximation) for $\mathcal{C}$. We begin with the following definition[5]:

Definition 5.1 *Two subspaces of* $\mathbb{R}^n$*, denoted by* X *and* Y*, are said to be ambient isotopic if there exists a continuous function* $H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ *satisfying:*
(1) $H(\cdot, 0)$ *is the identity,*
(2) $H(X, 1) = Y$*, and*
(3) $\forall t \in [0,1]$, $H(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ *is a homeomorphism.*
H *is an ambient isotopy between* X *and* Y.

In proving the isotopy, the foundation relied upon is the angular convergence, and the technique used here is the well known "push" [2]. It is sufficient here to consider a specialized type of push, which we will designate as a *median push*, as indicated in the next definition.

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[5] The definition given here is specialized to $\mathbb{R}^n$ as sufficient for the applications intended, but further generalizations are available [7].

Definition 5.2 *Assume that triangle $\triangle ABC$ has non-collinear vertices A, B and C. Push a vertex, say B, along the corresponding median of the triangle to the middle point of the side AC. We call this specific kind of "push", a median push*[6].

Lemma 5.1 *If a vertex P_j ($j \in \{1, \cdots, n-1\}$) of a polygon $P = (P_0, P_1, \cdots, P_n)$ in $\mathbb{R}^3$ undergoes a median push, then the total curvatures of the new open PL curves formed during the push remain the same or decrease, while the trace of the push for P_j remains within the triangle $\Delta P_{j-1}P_jP_{j+1}$. (This holds not only for the median push, but also for any push with a trace lying on the interior of a triangle indicated in Definition 5.2.)*

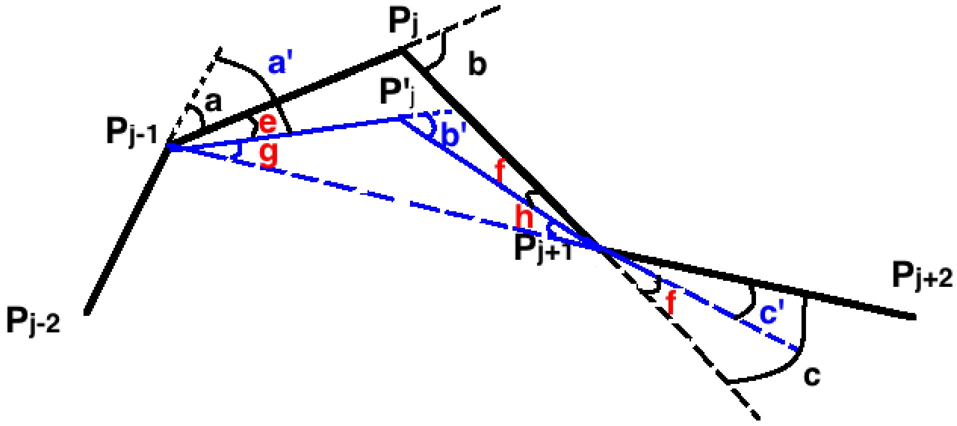

Figure 5: The change of exterior angles during a push

Proof: Consider Figure 5, which illustrates a non-planar polygonal curve in $\mathbb{R}^3$. For $\triangle P_{j-1}P_jP_{j+1}$, consider the median having P_j as an endpoint and denote an arbitrary interior point of that median by P'_j. Suppose that P_j is pushed to P'_j by the median push. Then the exterior angles affected by this push are a, b and c. The other angles stay unchanged. So we can compare the total curvatures by comparing $a+b+c$ and $a'+b'+c'$ where a', b' and c' are corresponding angles after the push.

It follows from the triangle inequality of spherical triangles [18] that

$$a' \leq a + e \textit{ and } c' \leq f + c,$$

and note that

$$b' = g + h \textit{ and } b = e + g + h + f,$$

where a', b' and c' are corresponding exterior angles to a, b and c after the push, and

$$e = \angle P_jP_{j-1}P_{j'}, \quad f = \angle P_jP_{j+1}P_{j'}, \quad g = \angle P_{j'}P_{j-1}P_{j+1} \textit{ and } h = \angle P_{j'}P_{j+1}P_{j-1}$$

---

[6] The paper [11] gives a similar result which compares the curvatures before and after a push but not those in between.

all of which are radians measured between 0 and π. So

$$a' + b' + c' \leq a + e + g + h + f + c \leq a + b + c$$

Therefore the total curvature remains the same or decreases during the median push. $\square$

After i iterations, there are 2^i sub-control polygons, denoted as $\mathbf{P}_j = (P_{j,0}, \dots, P_{j,n})$, $j = 1, \dots, 2^i$. Denote the control polygon as $K = \bigcup_{j=1,\dots 2^i} \mathbf{P}_j$. Let $L = \bigcup_{j=1,\dots 2^i} P_{j,0}P_{j,n}$ be the PL curve formed by connecting the cords from $P_{j,0}$ to $P_{j,n}$, for all $j = 1, \dots, 2^i$.

Theorem 5.1 *For* $\mathbf{C}$*, after sufficiently many subdivisions, the control polygon* K*, will be ambient isotopic to the corresponding* L*.*

Proof: We first choose a non-self-intersecting pipe surface $S_c(r)$ for $\mathbf{C}$ as in [10], where the hypotheses that $\mathbf{C}$ is simple, regular and C^2 are all invoked. Take sufficiently many (say I) subdivisions such that the convex hull of the control polygon lies inside the surface $S_{\mathcal{C}}(r)$ (Convergence in distance [15]) and each sub-control polygon has a total curvature less than $\frac{\pi}{2}$ (Theorem 3.1: Angular convergence). Lemma 4.4 implies that each sub-control polygon and the associated sub Bézier curve lie inside the corresponding pipe section . And the proof of Theorem 4.3 implies that the control polygon K is simple.

We use median pushes to move vertices of K onto L. Consider a median push on a vertex other than the end points of a sub-control polygon $\mathbf{P}_j$. Lemma 5.1 implies that the total curvature of $\mathbf{P}_j$ remains less than $\frac{\pi}{2}$ and $\mathbf{P}_j$ stays inside the corresponding pipe section during the push (Lemma 4.4), so the push does not yield self-intersection of $\mathbf{P}_j$ (Lemma 4.1) or pairwise intersections between $\mathbf{P}_j$ and any other sub-control polygons. Consequently the push does not yield self-intersections of K.

A sequence of median pushes will deform K to a sequence of PL curves $K_1, K_2, \cdots, L$. These specific pushes do not yield self-intersections of K, so this sequence induces ambient isotopies between each pair of the consecutive PL curves [1]. Moreover, these pushes leave the end points of all sub-control polygons as fixed points. Then the ambient isotopy between K and L follows from equivalence relation of ambient isotopy [9]. $\square$

Remark 5.2.1 *The proof of Theorem 5.1 relies upon equivalence relation of ambient isotopy. Another way of proving this could be defining an ambient isotopy directly between* K *and* L*, without relying on the equivalence relation. This would require pushes to yield neither the self-intersections of* K *nor the intersections between* K *and* L*. That proof would be more technical and an additional technique needed is called the 'simulation of simplicity'* [5].

Theorem 5.2 (Ambient isotopy) *Sufficiently many subdivisions will yield an ambient isotopic control polygon (PL approximation) for a Bézier curve* $\mathbf{C}$*.*

Proof: The PL curve L obtained by I subdivisions (in the proof of Theorem 5.1) is ambient isotopic to $\mathcal{C}$ [10]. By equivalence relation of ambient isotopy [9], Theorem 5.1 implies that K generated by the I subdivisions is ambient isotopic to $\mathcal{C}$. $\square$

6 Number of subdivisions needed

This section, again for ease of exposition, reverts back to the notation $\mathcal{C}$ for a single Bézier curve. The only subtlety to extend to a composite curve $\mathbf{C}$ is to ensure that the indicated bounds are taken over all individual curves forming the composite curve. Those details are merely tedious and are left to the interested reader.

We have already found an angular convergence rate. Based on the angular convergence rate and the convergence rate in distance, we shall find the speed for sub-control polygons being simple and furthermore the control polygon being simple and ambient isotopic to $\mathcal{C}$. Moreover a numerical lower bound of subdivision iterations to achieve these topological properties will be provided.

Definition 6.1 *As in Section 1.1, let P denote a control polygon of a Bézier curve, and let P_x denote an ordered list of all of x-coordinates of P (with similar meaning given to P_y for the y-coordinates and to P_z for the z-coordinates). Let*

$$\| \triangle_2 P_x \|_\infty = \max_{0<m<n} |P_{m-1,x} - 2P_{m,x} + P_{m+1,x}|$$

be the maximum absolute second difference of the x-coordinates of control points, (with similar meanings for the y and z coordinates) . Let

$$\mathcal{P} = (\| \triangle_2 P_x \|_\infty, \| \triangle_2 P_y \|_\infty, \| \triangle_2 P_x z \|_\infty),$$

(i.e.) a vector with 3 values. Let

$$||\Delta_2 P|| = ||\mathcal{P}||.$$

Definition 6.2 *The distance between a Bézier curve $\mathcal{C}$ and the control polygon $l(P,i)$ generated by i subdivisions is*

$$\| l(P,i) - \mathcal{C} \|_\infty = \max_{t\in[0,1]} ||l(P,i)(t) - \mathcal{C}(t)||.$$

Lemma 6.1 *The distance between the Bézier curve and its control polygon after ith-round subdivision is bounded by*

$$\frac{1}{2^{2i}} N_\infty(n) ||\Delta_2 P||, \tag{8}$$

where

$$N_\infty(n) = \frac{\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil}{2n}.$$

Proof: A published lemma [15, Lemma 6.2] proves this result restricted to scalar valued polynomials. We consider coordinate-wise and apply this result to the x, y, and z coordinates respectively, so that the distance of the x-coordinates of the Bézier curve and its control polygon after ith-round subdivision is bounded by

$$\frac{1}{2^{2i}} N_\infty(n) \| \Delta_2 P_x \|_\infty,$$

with similar expressions for the y and z coordinates. Taking the Euclidean norm of the indicated three x, y and z bounds yields Equation 8, an upper bound of the distance between the Bézier curve and its control polygon after the ith subdivision. $\square$

For convenience, denote the above bound in distance as:

$$B_{dist}(i) := \frac{1}{2^{2i}} N_\infty(n) ||\Delta_2 P||. \tag{9}$$

Lemma 6.2 *After i subdivision iterations, the distance between $l'(P,i)$ and $\mathcal{C}'$ is bounded by $B'_{dist}(i)$, where*

$$B'_{dist}(i) := \frac{1}{2^{2i}} N_\infty(n-1) ||\Delta_2 P'||, \tag{10}$$

and P' that consists of $n-1$ control points is the control polygon of $\mathcal{C}'$.

Proof: The derivative ($l'(P,i)$) of the control polygon for a Bézier curve is identical to the control polygon ($l(P',i)$) for the Bézier curve's first derivative by [13][Lemma 6]. That is,

$$l'(P,i) = l(P',i).$$

Since (Lemma 6.1)

$$\| \ l(P',i) - \mathcal{C}' \ \|_\infty = \max_{t \in [0,1]} ||l(P',i)(t) - \mathcal{C}'(t)|| \leq B'_{dist}(i),$$

we have

$$\| \ l'(P,i) - \mathcal{C}' \ \|_\infty \leq B'_{dist}(i). \tag{11}$$

$\square$

6.1 Number of subdivisions for small exterior angles

Assume θ is a small measure of angle between 0 and π. Consider the number of subdivisions that would generate a control polygon such that the measure α of each exterior angle satisfies

$$\alpha < \theta. \tag{12}$$

Recall the proof of angular convergence (Theorem 3.1). Consider two arbitrary consecutive first derivatives $u_i = l'(P,i)(t_m)$ and $v_i = l'(P,i)(t_{m-1})$ and the corresponding exterior angle α. Recall that in Section 3 we had:

$$1 - \cos(\alpha) \leq \frac{||u_i|| - ||v_i||}{||u_i||} + \frac{||v_i - u_i||}{||u_i||} \leq \frac{2||v_i - u_i||}{||u_i||}.$$

Recall the proof of Lemma 3.1 where Inequality 2 is:

$$||u_i - v_i||$$

$$\leq ||l'(P,i)(t_m) - \mathcal{C}'(t_m)|| + ||\mathcal{C}'(t_m) - \mathcal{C}'(t_{m-1})|| + ||\mathcal{C}'(t_{m-1}) - l'(P,i)(t_{m-1})||,$$

and Inequality 3 is

$$||\mathcal{C}'(t_m) - \mathcal{C}'(t_{m-1})|| \leq \sup_{[0,1]} ||\mathcal{C}''(t)|| \cdot |t_m - t_{m-1}| = \frac{\gamma}{n2^i},$$

where $\gamma = \sup_{[0,1]} ||\mathcal{C}''(t)||$. Combining the above inequalities yields

$$1 - \cos(\alpha) \leq \frac{2(||l'(P,i)(t_m) - \mathcal{C}'(t_m)|| + ||\mathcal{C}'(t_{m-1}) - l'(P,i)(t_{m-1})|| + \gamma/(n2^i))}{||u_i||}.$$

Using Lemma 6.2 we get

$$1 - \cos(\alpha) \leq \frac{2(2B'_{dist}(i) + \gamma/(n2^i))}{||u_i||}. \tag{13}$$

Let $\sigma = \min\{||\mathcal{C}'(t)|| : t \in [0,1]\}$. The regularity of $\mathcal{C}$ ensures that $\sigma > 0$ and the continuity of $\mathcal{C}'$ on the compact interval $[0,1]$ ensures that the minimum exists. Recall (in the proof of Theorem 3.1) that $u_i = l'(P,i)(t)$ for some $t \in [0,1]$, which will be denoted as $\hat{t}$. So it follows from Equation 11 that

$$||\mathcal{C}'(\hat{t})|| - ||u_i|| \leq B'_{dist}(i).$$

Solving the inequality we get

$$||u_i|| \geq ||\mathcal{C}'(\hat{t})|| - B'_{dist}(i) \geq \sigma - B'_{dist}(i).$$

In order to have $u_i \neq 0$, it is sufficient to perform enough subdivisions such that

$$||u_i|| \geq \sigma - B'_{dist}(i) > 0,$$

that is $B'_{dist}(i) < \sigma$. So by the definition (Equation 10) of $B'_{dist}(i)$ we derive,

$$\frac{1}{2^{2i}} N_\infty(n-1) \parallel \triangle_2 P' \parallel_\infty < \sigma.$$

Equivalently it suffices to have[7]

$$i > \frac{1}{2}\log(\frac{N_\infty(n-1) \parallel \triangle_2 P' \parallel_\infty}{\sigma}) = N_1. \tag{14}$$

It is worth noting that N_1 is a subdivision number depending on variables P', n and σ.

After the i subdivision iterations, whenever $i > N_1$, then $B'_{dist}(i) < B'_{dist}(N_1)$, because $B'_{dist}(i)$ is a strictly decreasing function (Equation 10). So it follows from Equation 13 that whenever $i > N_1$,

$$1 - \cos(\alpha) \leq \frac{2(2B'_{dist}(i) + \gamma/(n2^i))}{|\sigma - B'_{dist}(i)|} \leq \frac{2(2B'_{dist}(i) + \gamma/(n2^i))}{\sigma - B'_{dist}(N_1)}. \tag{15}$$

---

[7] Throughout this paper, we use log for $\log_2$.

Since $\alpha < \theta$ (Equation 12), it follows that $1 - cos(\alpha) < 1 - cos(\theta)$. Further consider

$$\frac{2(2B'_{dist}(i) + \gamma/(n2^i))}{\sigma - B'_{dist}(N_1)} < 1 - \cos(\theta),$$

that is

$$2B'_{dist}(i) + \frac{\gamma}{n2^i} < \frac{1}{2}(1 - \cos(\theta))(\sigma - B'_{dist}(N_1)).$$

We could solve the above inequality for i, but we can avoid this complicated computation by noting that $2B'_{dist}(i)$ is much smaller than $\frac{\gamma}{n2^i}$ when i is large. (Note that $\gamma = \sup_{[0,1]} ||\mathcal{C}''(t)||$. If $\gamma = 0$, then the Bézier curve would be a straight line segment. We exclude this trivial case and assume that $\gamma > 0$.)

So let $2B'_{dist}(i) < \frac{\gamma}{n2^i}$, that is

$$i > \log(\frac{2nN_\infty(n) \parallel \triangle_2 P \parallel_\infty}{\gamma}) = N_2. \tag{16}$$

Then we consider

$$2\frac{\gamma}{n2^i} < \frac{1}{2}(1 - \cos(\theta))(\sigma - B'_{dist}(N_1)). \tag{17}$$

Solve Inequality 17 and get

$$\frac{1}{2^i} < \frac{n}{4\gamma}(1 - \cos(\theta))(\sigma - B'_{dist}(N_1)).$$

Denote the right hand side of the above inequality as a function $f(\theta)$, that is

$$f(\theta) = \frac{n}{4\gamma}(1 - \cos(\theta))(\sigma - B'_{dist}(N_1)). \tag{18}$$

Then, we have

$$i > \log(\frac{1}{f(\theta)}). \tag{19}$$

Theorem 6.1 *Given any $\theta > 0$, there exists an integer $N(\theta)$, such that each exterior angle is less than θ for $i > N(\theta)$. Furthermore, there is an explicit closed formula to compute $N(\theta)$.*

Proof: Let

$$N(\theta) = \max\{N_1, N_2, \log(\frac{1}{f(\theta)})\} \tag{20}$$

where N_1, N_2 and $f(\theta)$ are given by Equations 14 16 and 18 respectively.

It is worth to note that N is a logarithm depending on several parameters such as γ, σ, $N_\infty(n)$ and $\triangle_2 P'$ as well as an upper bound variable θ.

6.2 Number of subdivisions for simple control polygons

For a Bézier curve $\mathcal{C}$ of degree 2 or 1, the control polygon is trivially simple and ambient isotopic to $\mathcal{C}$, provided the regularity of $\mathcal{C}$. So, from now on, we consider $n > 2$.

Given any $\theta > 0$, there exists an integer $N(\theta)$, such that each exterior angle is less than θ after $N(\theta)$. Furthermore, there is an explicit closed formula to compute $N(\theta)$.

Theorem 6.2 *There exists a positive integer, N, such that after the Nth subdivision, each sub-control polygon will be simple.*

Proof: Let $N = N(\frac{\pi}{n-1})$, where $n > 2$, and $N(\frac{\pi}{n-1})$ is defined in the proof of Theorem 6.1. Then after N subdivisions, each exterior angle is less than $\frac{\pi}{n-1}$ and each sub-control polygon has a $n-1$ exterior angles. So the total curvature of each sub-control polygon is less than π. Lemma 4.1 implies that this is a sufficient condition for each sub-control polygon being simple. □

As at the end of Section 4, consider a non-self-intersecting pipe surface of constant radius r about $\mathcal{C}$ [10]. While existence of sufficiently many iterations for the control polygon to fit inside the pipe has been established, it remains of interest to bound the number of subdivisions that are sufficient for this containment. The control polygon lies inside the pipe if the distance between the Bézier curve and the control polygon is less than r. By Lemma 6.1, it suffices to have

$$B_{dist}(i) := \frac{1}{2^{2i}} N_\infty(n) ||\Delta_2 P|| < r. \tag{21}$$

So

$$i > \frac{1}{2} \log\left(\frac{N_\infty(n)||\Delta_2 P||}{r}\right) = N'. \tag{22}$$

While Theorem 6.2 addresses each sub-control polygon, it is of interest to ensure that the union of all these sub-control polygons is also simple. In Theorem 6.3, that union is the 'control polygon', as the result of multiple subdivisions.

Theorem 6.3 *There exists a positive integer N, such that after the Nth subdivision, the control polygon will be simple. Furthermore, there is an explicit closed formula to compute N.*

Proof: Take $N \geq \max\{N(\frac{\pi}{2(n-1)}), N'\}$, where $N(\theta)$ is defined in Equations 20 and N' is given by Equations 22. Then after the Nth subdivision, the control polygon will be simple. Inequality 21 implies that the control polygon generated after the Nth subdivision lies inside the pipe. The inequality $N \geq N(\frac{\pi}{2(n-1)})$ ensures that the total curvature of its each sub-control polygon is less than $\frac{\pi}{2}$. These two conditions are sufficient conditions for the control polygon being simple (The proof of Theorem 4.3). □

6.3 Number of subdivisions for ambient isotopy

Now we consider how many subdivisions are needed to generate a control polygon ambient isotopic to $\mathcal{C}$. First the small total curvatures are needed. This was taken care of by taking $N(\frac{\pi}{2(n-1)})$ (Equation 20) subdivisions. Besides, recall, from the previous section, that the convex hull of the control polygon should also be contained within a non-self-intersecting pipe surface $S_c(r)$. A stronger, but convenient condition, is to ensure that the convex hull of each sub-control polygon is contained within the pipe.

The following lemma from the paper [8] indicates the distance of any two consecutive vertices of control polygons converges to 0 under subdivision.

Lemma 6.3 [8, Lemma 2.5] *For any two consecutive vertices R and Q of the corresponding control polygon after i subdivision iterations we have*

$$|R-Q| \leq (\frac{M}{2^i})$$

where $M = \max_{j\in\{1,2,\cdots,n\}}\{||P_j - P_{j-1}||\}$,$P_j$ and P_{j-1} are two connected control points of the initial control polygon before performing subdivision.

Lemma 6.4 *The convex hulls will fit inside the pipe surface $S_c(r)$ after the ith subdivision, where i is large enough such that $\frac{Mn}{2^i} < \frac{r}{2}$. That is*

$$i > \log(\frac{2Mn}{r}) = \hat{N}. \tag{23}$$

Proof: Consider the sphere with the center to be the initial control point of the sub-control polygon and radius to be $\frac{r}{2}$. Using Lemma 6.3 we get that the distance between the center and any vertex of this sub-control polygon is less than $\frac{Mn}{2^i}$ and hence less than $\frac{r}{2}$ by the hypothesis. So the sphere contains all of the vertices and furthermore its convex hull. By the convex hull property [6] of $\mathcal{C}$, the corresponding sub-curve of $\mathcal{C}$ is contained in the sphere. So the maximal distance between the sub-curve and the boundary of the sphere is less than r. Since the distance between the sub-curve and $S_c(r)$ is r, the sphere lies inside $S_c(r)$ and consequently the convex hull fits inside $S_c(r)$. □

Take $N \geq max\{N(\frac{\pi}{2(n-1)}), N'\}$, where $N(\theta)$ is defined in Equations 20 and N' is given by Equations 22.

Theorem 6.4 *There exists a positive integer N, such that after the Nth subdivision, the control polygon will be ambient isotopic to the Bézier curve $\mathcal{C}$. Furthermore, there is an explicit closed formula to compute N.*

Proof: Take $N \geq max\{N(\frac{\pi}{2(n-1)}), \hat{N}\}$, where $N(\theta)$ is defined in Equations 20 and $\hat{N}$ is given by Equations 23. Then after the Nth subdivision, the control polygon will be ambient isotopic to the Bézier curve $\mathcal{C}$, as follows from the above Lemma 6.4 and the ambient isotopy Theorem 5.2. □

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