Isotopy Convergence Theorem

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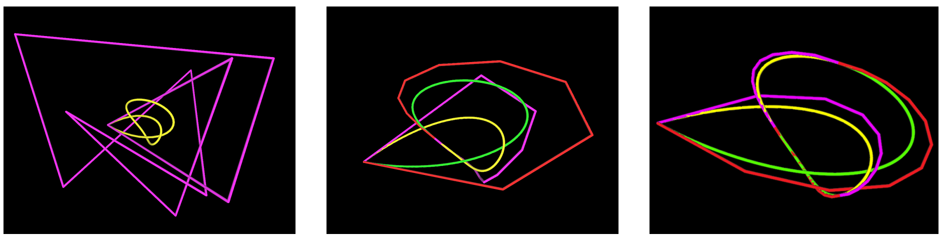
It is of contemporary interest to designate particular topological aspects that should be preserve under space curve approximation. Particularly in computer graphics, visualization, and molecular animation, preservation of knot structure is of increasing demand. But ambient isotopy is the equivalence relation in knot theory. This leads us to designate convergence criteria that ensures ambient isotopic approximation.



(a) Unknot VS. Knot (b) An intermediate step (c) Knot VS. Knot

Figure 1: Ambient isotopic approximation

Figure 1(a) is shows an example of a knotted spline defined by an unknotted PL curve. Then the standard algorithm called subdivision [1] is applied to generate new PL curves as Figure 1(b) and 1(c) show. The sequence of PL curves generated by subdivision will eventually have the same knot type as the given spline such as that in Figure 1(c).

Theorem 1 (Isotopy convergence theorem) *If the sequence of curves $\{C_i\}_1^\infty$ converges to curve*[1] *$\mathcal{C}$ in both distance and total curvature, then there exists an N such that C_i is ambient isotopic to $\mathcal{C}$ for $i \geq N$.*

Definition 1 [2] *A closed PL curve with vertices $\alpha_1, \alpha_2, \cdots, \alpha_m$ is said to be inscribed in another curve $C(t)$ if there is a sequence $\{t_i\}_1^m$ of parameter values such that $\alpha_i = C(t_i)$ for $i = 1, 2, \cdots, m$.*

A special case of Theorem 1 is using inscribed PL curves to approximate a given curve. In this case, the sequence of inscribed PL curves converges to the given curve in both distance and total curvature [2]. Then we get:

Theorem 2 *A sequence $\{L_i\}_1^\infty$ of PL curves that approximate and inscribed in $\mathcal{C}$ will eventually be ambient isotopic to $\mathcal{C}$.*

The Isotopy convergence theorem formulates a convergence criteria to show the same knot type between knots. In particular, based on my proof of Theorem 2, an algorithm can be developed such that it takes a smooth knot as input and picks finitely many points on it to form an ambient isotopic PL knot. This provides advantages. For instance, the knot type of the smooth knot is determined by the set of finitely many points.

A future research direction might be some investigation using the Isotopy convergence theorem in knot theory. Besides, practical applications might be explored.

[1]The symbol $\mathcal{C}$ is used to denote a compact, regular and C^2 space curve.

References

[1] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design.* Academic Press, Inc., San Diego, CA, 1990.

[2] J. W. Milnor. On the total curvature of knots. *Ann. Math.*, 52:248–257, 1950.