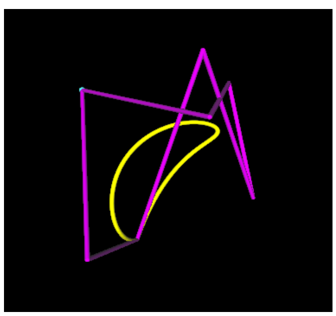
Ambient Isotopy under Subdivision

J.Li T.J.Peters J.A.Roulier

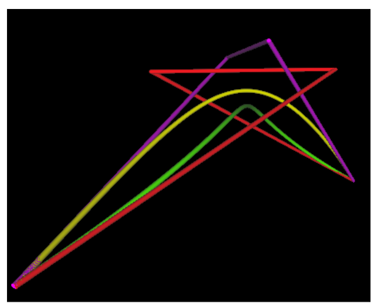
Definition 1 (Ambient isotopy) *An ambient isotopy between two subspaces X and Y of $\mathbb{R}^n$, is a continuous function $H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ satisfying: (1) $H(\cdot, 0)$ is the identity, (2) $H(X, 1) = Y$ and (3) $H(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism for $\forall t \in [0,1]$. If H exists, then X and Y are called ambient isotopic.*

Ambient isotopy was selected as a more appropriate model for time varying topological characteristics than the standard topological equivalence given by homeomorphism. Preservation of ambient isotopy is both an important theoretical and practical concern, but the potential of the applications directs us to move beyond pure existence theorems into a specific class of curves, Bézier curves, which are parametrized polynomials widely used in computer-aided geometric design. A Bézier curve with degree n is determined by a PL (piecewise linear) curve of n segments, called a control polygon.

1 Topological differences



(a) Not homeomorphic



(b) Not ambient isotopic

Figure 1: Topological differences

There may be substantial topological differences between a Bézier curve and its control polygon. Figure 1(a) shows a simple (non-self-intersecting) Bézier curve and a self-intersecting control polygon. They are not homeomorphic. Figure 1(b) displays a unknotted Bézier curve and a knotted control polygon. Even though homeomorphic, they are not ambient isotopic.

2 Ambient isotopy under subdivision

Subdivision recursively generates new control polygons that more closely approximate a Bézier curve under Hausdorff distance [3], as Figure 2 shows a new control polygon generated by a subdivision performed on Figure 1(b). The new control polygon in Figure 2 is

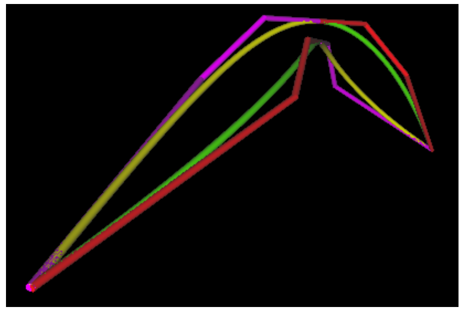


Figure 2: Ambient isotopy

unknotted and further ambient isotopic [1]. This naturally raises the question of whether this subdivision process will always produce an ambient isotopic PL approximation.
The symbol $\mathcal{B}$ will denote a simple C^2, regular Bézier curve of arbitrary degree n in $\mathbb{R}^3$.

Theorem 1 (Ambient isotopy) *Sufficiently many subdivisions yield an ambient isotopic approximation for a Bézier curve* $\mathcal{B}$*.*

The result here extends a previous theorem [2] which was restricted to Bézier curves of low degree (less than 4), where a crucial unknotting condition was trivial. Deriving a comparable unknotting condition for higher degrees entailed significant new arguments that rely on the following theorem that was previously unknown to the best of our knowledge.

Theorem 2 (Angular convergence) *For a Bézier curve* $\mathcal{B}$*, the exterior angles of the control polygons generated by subdivision converge to* 0 *at a rate of* $O(\sqrt{\frac{1}{2^i}})$*.*

Besides, the angular convergence rate determines an exponential speed for the control polygons converging to be ambient isotopic to a Bézier curve. Furthermore, a sufficient number of subdivision iterations to achieve ambient isotopy was found.

3 Applications

This theory is being applied to visualizing molecular simulations with Bézier curves on high performance computing architectures. Precisely, when Bézier curves are used to model macro-molecules in order to perform dynamic visualization of the molecular writhing and perturbation, a fundamental requirement is the ambient isotopic equivalence.

References

[1] J. Bisceglio, T. J. Peters, J. A. Roulier, and C. H. Sequin. Unknots with highly knotted control polygons. *Computer Aided Geometric Design*, 28(3):212–214, 2011.

[2] E. L. F. Moore, T. J. Peters, and J. A. Roulier. Preserving computational topology by subdivision of quadratic and cubic Bézier curves. *Computing*, 79(2-4):317–323, 2007.

[3] L. Piegl and W. Tiller. *The NURBS Book*. Springer, New York, NY, 2nd edition, 1997.