Risk Theory Prelims for Actuarial Students
Monday, 16 January 2017
MONT 214, 9:00 am - 1:00 pm

Instructions:

1. There are five (5) questions here and you are to answer all five. Each question is worth 20 points.
2. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
3. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

Question No. 1:

Let X be a random variable with distribution function F that is continuous and has support $(0, \infty)$. The mean excess loss function of X is defined by

$$e(x) = \mathrm{E}(X - x | X > x).$$

(a) Show that $e(\cdot)$ satisfies

$$e(x) = \frac{1}{1 - F(x)} \int_x^\infty [1 - F(y)] dy$$

(b) Show that the distribution of X is uniquely determined by its mean excess loss function by proving that

$$F(x) = 1 - \frac{e(0)}{e(x)} \exp\left\{ -\int_0^x [e(y)]^{-1} dy \right\},$$

for $x > 0$.

(c) Suppose the distribution function of X has the representation

$$F(x) = 1 - \theta^\alpha x^{-\alpha}, \quad \text{for } x > \theta,$$

for $\theta > 0$ and $\alpha > 0$. Calculate the mean excess loss function of X for $\alpha > 1$. Why do we need the condition that $\alpha > 1$? Verify that $e(x) \to \infty$ as $x \to 0$.

Question No. 2:

Define the total claims S as $S = X_1 + X_2 + \cdots + X_N$ where the claim amount X_i, for $i = 1, 2, \cdots$, has integer-valued non-negative domain and the total number of claims N is a member of the $(a, b, 0)$ class of discrete distributions. Assume claim amount X_i are independent random variables with common density function $f_X(\cdot)$.

(a) Prove that the probability generating function (pgf) of S can be expressed as

$$P_S(s) = P_N[P_X(s)],$$

where $P_N(\cdot)$ and $P_X(\cdot)$ as the pgf's of N and X, respectively.

(b) Prove the so-called Panjer's recursion formula which states that the density function of S can be expressed as

$$f_S(s) = \frac{1}{1 - af_X(0)} \sum_{h=1}^{s} \left(a + \frac{bh}{s}\right) f_X(h) f_S(s-h).$$

(c) Show that for $s = 0$, $f_S(0) = \Pr(N = 0)$ if $f_X(0) = 0$ and $f_S(0) = P_N[f_X(0)]$ if $f_X(0) > 0$.

Question No. 3:

Individual loss amount X follows a two-parameter Pareto distribution with mean 2 and variance 8. An insurance policy on X has a deductible amount of 1 and a policy limit of 5 per loss. Assume loss amount increased due to inflation by 5% uniformly.

(a) Calculate the expected value of claims per loss after the inflation.

(b) Calculate the variance of claims per loss after the inflation.

Question No. 4:

For a risk X, denote the value-at-risk at the $100p\%$ level by $\mathrm{VaR}_p(X) = \pi_p$ and the tail value-at-risk at the $100p\%$ level by $\mathrm{TVaR}_p(X)$.

(a) Demonstrate that each of the following expressions are true:

$$\mathrm{TVaR}_p(X) = \frac{1}{1-p} \int_p^1 \pi_u du$$

$$\mathrm{TVaR}_p(X) = \pi_p + e(\pi_p)$$

$$\mathrm{TVaR}_p(X) = \pi_p + \frac{1}{1-p}[\mathrm{E}(X) - \mathrm{E}(X \wedge \pi_p)]$$

(b) Let X_1 and X_2 be two independent exponential distributions with means 0.5 and 1, respectively. Define $S = X_1 + X_2$. Calculate $\mathrm{VaR}_{0.95}(S)$.

Question No. 5:

An insurance company has a surplus process with a compound Poisson claims process.
You are given that:

- relative security loading is 60%; and
- claim amount distribution is a mixture of exponentials:

$$p(x) = 3e^{-4x} + \frac{1}{2}e^{-2x}, \text{ for } x \geq 0.$$

Obtain an expression for the probability of ruin, $\psi(u)$, using Tijms' approximation.

APPENDIX

A random variable X is said to have a two-parameter Pareto distribution if its density has the form

$$f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, \quad \text{for } x > 0.$$

This distribution satisfies the following:

$$\mathrm{E}[(X \wedge x)^k] = \frac{\theta^k \Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)} \beta[k+1, \alpha-k; x/(x+\theta)] + x^k \left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{for any } k.$$

A discrete random variable N is said to belong to the $(a, b, 0)$ class of distributions if it satisfies the relation

$$\Pr(N = k) = p_k = \left(a + \frac{b}{k}\right) \cdot p_{k-1}, \quad \text{for } k = 1, 2, \ldots,$$

for some constants a and b. The initial value p_0 is determined so that $\sum_{k=0}^{\infty} p_k = 1$.