University of Connecticut
Department of Mathematics
Preliminary Examination - Risk Theory (Math 395)
August 22, 2008, 9 a.m.

There are 5 questions. Show all of your calculations and give the reasons that justify your steps, although you do not need to give formal proofs for your results. A summary of key formulas for a variety of distributions is attached. You may use any hand-held calculator. There are 3 hours for this examination. Mark your candidate number clearly on each blue book or page that you submit, but do not identify yourself in any other way.

1. The Hermite polynomials are defined to be the polynomials $H_n(x)$ that make the following expression true where $\phi(x) = e^{-\frac{1}{2}x^2}$ is the standard normal density:
$$\phi^{(n)}(x) = H_n(x)\phi(x).$$
Use Faá's Formula to help you write down the general formula, including the coefficients, for the polynomial $H_n(x)$.

2. A certain accident probability follows a Poisson distribution for any particular exposed risk in a population of 1,000,000 total exposures. Different exposed risks are characterized by different Poisson frequency parameters λ. The frequency parameter λ is distributed across the entire population as a sum of ten independent *gamma* random variables, all having the same β (or θ) parameters. One of them has α parameter $\alpha = 1$, three of them have $\alpha = 2$, and six of them have $\alpha = 3$. Finally, suppose that the population of exposures increases to 1,500,000 but does so in a way that the aggregate accident probability distribution is proportionally unchanged. What are the mean, variance and third central moment of the accident frequency for the new population of 1,500,000 exposures?

3. A surplus process is defined by $u(t) = u + ct - S(t)$ where $S(t)$ is compound Poisson with $\lambda = 400$ and individual loss distribution $p(x) = 2e^{-2x}$. What is the largest premium accumulation rate c for which the probability of ruin $\psi(u)$ is completely independent of the initial surplus u? Be sure to explain why your answer is the correct one.

4. Let $S(t) = X_1 + ... + X_{N(t)}$ where $N(t)$ is Poisson with frequency $20t$ and the X's are are independently and identically distributed with the property that the conditional distribution of $S(t)$, conditional on $N(t) = N^*$, is a *gamma* distribution with parameter $\alpha = N^*$ and mean $3N^*$ for each integer N^*. Let $L = \max_{t \geq 0} \{(S(t) - 66t)_+\}$ be the maximum aggregate loss random variable with premium rate $c = 66$. Express $L = K_1 + ... + K_M$ where M is a random counting variable and the K's are i.i.d. Approximate K by rounding using a discrete distribution with whole integer units. Calculate the resulting approximate values for

   (a) the probability $\psi(4)$ of ruin from a starting surplus of 4.

   (b) the expected value $\mathbb{E}[(L-4)_+ | L > 4]$ for the largest excess of accumulated losses over the accumulated premium plus starting surplus of 4, contingent upon ruin ever occuring.

5. Individual loss amounts (ground up) this year follow a Weibull distribution with mean θ and standard deviation θ. Next year you confidently expect loss amounts to inflate by 10% uniformly across the board. What will be the standard deviation next year for loss amounts that are subjected to a 100 deductible per loss, but with losses prior to the deductible limited to $1,000$ per loss. Give the answer for the "per loss" variable, not the "per payment" variable. Assume $\theta = 300$.

Appendix A

An Inventory of Continuous Distributions

A.1 Introduction

The incomplete gamma function is given by

$$\Gamma(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\, dt, \qquad \alpha > 0,\ x > 0$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt, \qquad \alpha > 0.$$

When $\alpha \leq 0$ the integral does not exist. In that case, define

$$\Gamma(\alpha) G(\alpha; x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt, \qquad x > 0.$$

Integration by parts produces the relationship

$$\Gamma(\alpha) G(\alpha; x) = -\frac{x^\alpha e^{-x}}{\alpha} + \frac{\Gamma(\alpha+1)}{\alpha} G(\alpha+1; x)$$

which allows for recursive calculation because for $\alpha > 0$, $\Gamma(\alpha)G(\alpha; x) = \Gamma(\alpha)[1 - \Gamma(\alpha; x)]$.

The incomplete beta function is given by

$$\beta(a, b; x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1} (1-t)^{b-1}\, dt, \qquad a > 0,\ b > 0,\ 0 < x < 1.$$

A.2 Transformed beta family

A.2.3 Two-parameter distributions

A.2.3.1 Pareto—α, θ

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \qquad F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha \\
E[X^k] &= \frac{\theta^k \Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha \\
E[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
E[X \wedge x] &= \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1 \\
E[X \wedge x] &= -\theta \log\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1 \\
E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1, \alpha-k; x/(x+\theta)] + x^k \left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{all } k \\
\text{mode} &= 0
\end{aligned}
$$

A.2.3.2 Inverse Pareto—τ, θ

$$
\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} \qquad F(x) = \left(\frac{x}{x+\theta}\right)^\tau \\
E[X^k] &= \frac{\theta^k \Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1 \\
E[X^k] &= \frac{\theta^k (-k)!}{(\tau-1)\cdots(\tau+k)}, \quad \text{if } k \text{ is a negative integer} \\
E[(X \wedge x)^k] &= \theta^k \tau \int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k} dy + x^k \left[1 - \left(\frac{x}{x+\theta}\right)^\tau\right], \quad k > -\tau \\
\text{mode} &= \theta\frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.3 Loglogistic—γ, θ

$$
\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2} \qquad F(x) = u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
E[X^k] &= \theta^k \Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma \\
E[(X \wedge x)^k] &= \theta^k \Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma \\
\text{mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.4 Paralogistic—α, θ

This is a Burr distribution with $\gamma = \alpha$.

$$\begin{aligned} f(x) &= \frac{\alpha^2 (x/\theta)^\alpha}{x[1+(x/\theta)^\alpha]^{\alpha+1}} \qquad F(x) = 1 - u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\alpha} \\ E[X^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha - k/\alpha)}{\Gamma(\alpha)}, \quad -\alpha < k < \alpha^2 \\ E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha - k/\alpha)}{\Gamma(\alpha)} \beta(1+k/\alpha, \alpha - k/\alpha; 1-u) + x^k u^\alpha, \quad k > -\alpha \\ \text{mode} &= \theta \left(\frac{\alpha - 1}{\alpha^2 + 1} \right)^{1/\alpha}, \quad \alpha > 1, \text{ else } 0 \end{aligned}$$

A.2.3.5 Inverse paralogistic—τ, θ

This is an inverse Burr distribution with $\gamma = \tau$.

$$\begin{aligned} f(x) &= \frac{\tau^2 (x/\theta)^{\tau^2}}{x[1+(x/\theta)^\tau]^{\tau+1}} \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\tau}{1+(x/\theta)^\tau} \\ E[X^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1 - k/\tau)}{\Gamma(\tau)}, \quad -\tau^2 < k < \tau \\ E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1 - k/\tau)}{\Gamma(\tau)} \beta(\tau + k/\tau, 1 - k/\tau; u) + x^k [1 - u^\tau], \quad k > -\tau^2 \\ \text{mode} &= \theta (\tau - 1)^{1/\tau}, \quad \tau > 1, \text{ else } 0 \end{aligned}$$

A.3 Transformed gamma family

A.3.2 Two-parameter distributions

A.3.2.1 Gamma—α, θ

$$\begin{aligned} f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)} \qquad F(x) = \Gamma(\alpha; x/\theta) \\ M(t) &= (1 - \theta t)^{-\alpha} \qquad E[X^k] = \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)}, \quad k > -\alpha \\ E[X^k] &= \theta^k (\alpha + k - 1) \cdots \alpha, \quad \text{if } k \text{ is an integer} \end{aligned}$$

$$\begin{aligned} E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)} \Gamma(\alpha + k; x/\theta) + x^k [1 - \Gamma(\alpha; x/\theta)], \quad k > -\alpha \\ &= \alpha(\alpha+1) \cdots (\alpha + k - 1) \theta^k \Gamma(\alpha + k; x/\theta) + x^k [1 - \Gamma(\alpha; x/\theta)], \quad k \text{ an integer} \\ \text{mode} &= \theta(\alpha - 1), \quad \alpha > 1, \text{ else } 0 \end{aligned}$$

A.3.2.2 Inverse gamma—α, θ

$$
\begin{aligned}
f(x) &= \frac{(\theta/x)^\alpha e^{-\theta/x}}{x\Gamma(\alpha)} \qquad F(x) = 1 - \Gamma(\alpha; \theta/x) \\
E[X^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}, \quad k < \alpha \qquad E[X^k] = \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}[1 - \Gamma(\alpha - k; \theta/x)] + x^k \Gamma(\alpha; \theta/x) \\
&= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)} G(\alpha - k; \theta/x) + x^k \Gamma(\alpha; \theta/x), \text{ all } k \\
&= \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)} G(\alpha - k; \theta/x) + x^k \Gamma(\alpha; \theta/x), \ k \text{ is an integer} \\
\text{mode} &= \theta/(\alpha+1)
\end{aligned}
$$

A.3.2.3 Weibull—θ, τ

$$
\begin{aligned}
f(x) &= \frac{\tau (x/\theta)^\tau e^{-(x/\theta)^\tau}}{x} \qquad F(x) = 1 - e^{-(x/\theta)^\tau} \\
E[X^k] &= \theta^k \Gamma(1 + k/\tau), \quad k > -\tau \\
E[(X \wedge x)^k] &= \theta^k \Gamma(1 + k/\tau) \Gamma[1 + k/\tau; (x/\theta)^\tau] + x^k e^{-(x/\theta)^\tau}, \quad k > -\tau \\
\text{mode} &= \theta \left(\frac{\tau - 1}{\tau} \right)^{1/\tau}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.3.2.4 Inverse Weibull—θ, τ

$$
\begin{aligned}
f(x) &= \frac{\tau (\theta/x)^\tau e^{-(\theta/x)^\tau}}{x} \qquad F(x) = e^{-(\theta/x)^\tau} \\
E[X^k] &= \theta^k \Gamma(1 - k/\tau), \quad k < \tau \\
E[(X \wedge x)^k] &= \theta^k \Gamma(1 - k/\tau)\{1 - \Gamma[1 - k/\tau; (\theta/x)^\tau]\} + x^k \left[1 - e^{-(\theta/x)^\tau}\right], \quad \text{all } k \\
&= \theta^k \Gamma(1 - k/\tau) G[1 - k/\tau; (\theta/x)^\tau] + x^k \left[1 - e^{-(\theta/x)^\tau}\right] \\
\text{mode} &= \theta \left(\frac{\tau}{\tau + 1} \right)^{1/\tau}
\end{aligned}
$$

A.3.3 One-parameter distributions

A.3.3.1 Exponential—θ

$$
\begin{aligned}
f(x) &= \frac{e^{-x/\theta}}{\theta} \qquad F(x) = 1 - e^{-x/\theta} \\
M(t) &= (1-\theta t)^{-1} \qquad E[X^k] = \theta^k \Gamma(k+1), \quad k > -1 \\
E[X^k] &= \theta^k k!, \quad \text{if } k \text{ is an integer} \\
E[X \wedge x] &= \theta(1 - e^{-x/\theta}) \\
E[(X \wedge x)^k] &= \theta^k \Gamma(k+1)\Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k > -1 \\
&= \theta^k k! \Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k \text{ an integer} \\
\text{mode} &= 0
\end{aligned}
$$

A.3.3.2 Inverse exponential—θ

$$
\begin{aligned}
f(x) &= \frac{\theta e^{-\theta/x}}{x^2} \qquad F(x) = e^{-\theta/x} \\
E[X^k] &= \theta^k \Gamma(1-k), \quad k < 1 \\
E[(X \wedge x)^k] &= \theta^k \Gamma(1-k) G(1-k; \theta/x) + x^k (1 - e^{-\theta/x}), \quad \text{all } k \\
\text{mode} &= \theta/2
\end{aligned}
$$

A.4 Other distributions

A.4.1.1 Lognormal—μ,σ (μ can be negative)

$$
\begin{aligned}
f(x) &= \frac{1}{x\sigma\sqrt{2\pi}} \exp(-z^2/2) = \phi(z)/(\sigma x), \quad z = \frac{\log x - \mu}{\sigma} \qquad F(x) = \Phi(z) \\
E[X^k] &= \exp(k\mu + k^2\sigma^2/2) \\
E[(X \wedge x)^k] &= \exp(k\mu + k^2\sigma^2/2)\Phi\left(\frac{\log x - \mu - k\sigma^2}{\sigma}\right) + x^k[1 - F(x)] \\
\text{mode} &= \exp(\mu - \sigma^2)
\end{aligned}
$$

A.4.1.2 Inverse Gaussian—μ, θ

$$
\begin{aligned}
f(x) &= \left(\frac{\theta}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\theta z^2}{2x}\right\}, \quad z = \frac{x-\mu}{\mu} \\
F(x) &= \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] + \exp(2\theta/\mu)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \quad y = \frac{x+\mu}{\mu} \\
M(t) &= \exp\left[\frac{\theta}{\mu}\left(1 - \sqrt{1 - \frac{2t\mu^2}{\theta}}\right)\right] \qquad E[X] = \mu, \quad Var[X] = \mu^3/\theta \\
E[X \wedge x] &= x - \mu z \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] - \mu y \exp(2\theta/\mu)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right]
\end{aligned}
$$

A.4.1.3 Single-parameter Pareto—α, θ

$$f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \quad x > \theta \qquad F(x) = 1 - (\theta/x)^\alpha, \quad x > \theta$$

$$E[X^k] = \frac{\alpha\theta^k}{\alpha - k}, \quad k < \alpha \qquad E[(X \wedge x)^k] = \frac{\alpha\theta^k}{\alpha - k} - \frac{k\theta^\alpha}{(\alpha - k)x^{\alpha-k}}$$

$$\text{mode} = \theta$$

Note: Although there appears to be two parameters, only α is a true parameter. The value of θ must be set in advance.

A.5 Distributions with finite support

For these two distributions, the scale parameter θ is assumed known.

A.5.1.1 Generalized beta—a, b, θ, τ

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{\tau}{x}, \quad 0 < x < \theta, \quad u = (x/\theta)^\tau$$

$$F(x) = \beta(a, b; u)$$

$$E[X^k] = \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)}, \quad k > -a\tau$$

$$E[(X \wedge x)^k] = \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)} \beta(a+k/\tau, b; u) + x^k[1 - \beta(a, b; u)]$$

A.5.1.2 beta—a, b, θ

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{1}{x}, \quad 0 < x < \theta, \quad u = x/\theta$$

$$F(x) = \beta(a, b; u)$$

$$E[X^k] = \frac{\theta^k \Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)}, \quad k > -a$$

$$E[X^k] = \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)}, \quad \text{if } k \text{ is an integer}$$

$$E[(X \wedge x)^k] = \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)} \beta(a+k, b; u) + x^k[1 - \beta(a, b; u)]$$

Appendix B

An Inventory of Discrete Distributions

B.2 The $(a,b,0)$ class

B.2.1.1 Poisson—λ

$$\begin{aligned} p_0 &= e^{-\lambda}, \quad a=0, \quad b=\lambda \qquad p_k = \frac{e^{-\lambda}\lambda^k}{k!} \\ E[N] &= \lambda, \quad Var[N]=\lambda \qquad P(z)=e^{\lambda(z-1)} \end{aligned}$$

B.2.1.2 Geometric—β

$$\begin{aligned} p_0 &= 1/(1+\beta), \quad a=\beta/(1+\beta), \quad b=0 \qquad p_k = \frac{\beta^k}{(1+\beta)^{k+1}} \\ E[N] &= \beta, \quad Var[N]=\beta(1+\beta) \qquad P(z)=[1-\beta(z-1)]^{-1} \end{aligned}$$

This is a special case of the negative binomial with $r=1$.

B.2.1.3 Binomial—$q, m, (0<q<1,\ m$ an integer)

$$\begin{aligned} p_0 &= (1-q)^m, \quad a=-q/(1-q), \quad b=(m+1)q/(1-q) \\ p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k=0,1,\ldots,m \\ E[N] &= mq, \quad Var[N]=mq(1-q) \qquad P(z)=[1+q(z-1)]^m \end{aligned}$$

B.2.1.4 Negative binomial—β, r

$$\begin{aligned} p_0 &= (1+\beta)^{-r}, \quad a=\beta/(1+\beta), \quad b=(r-1)\beta/(1+\beta) \\ p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \\ E[N] &= r\beta, \quad Var[N]=r\beta(1+\beta) \qquad P(z)=[1-\beta(z-1)]^{-r} \end{aligned}$$