

University of Connecticut
Department of Mathematics
Preliminary Exam – Risk Theory Section (Math 395)
August 25, 2004, 9AM

Show your calculations and give the reasons that justify your steps, although you need formally to prove your results only for problem number 4. You may refer to a summary of key formulas for a variety of distributions that is attached to this examination. You may use any hand-held calculator. There are 3 hours for the exam.

Make the following assumptions for problems 1, 2, and 3: A surplus process is defined by $u(t) = u + t - S(t)$ where $S(t)$ is a compound Poisson process with $\lambda = 3$ and individual claim distribution $p(x) = \frac{1}{3}e^{-3x} + \frac{16}{3}e^{-6x}$. The premium accumulation rate is $c = 1$.

1. Find the value of the safety loading θ .
2. Write an exact formula (with no unknown coefficients) for the moment generating function $M_X(r)$ of the individual claim distribution.
3. Write an exact formula (with no unknown coefficients) for the probability of ruin $\psi(u)$ as a function of the initial surplus u .

4. Prove that $\Gamma(n+1; x) = 1 - e^{-x} \sum_{j=0}^n \frac{x^j}{j!}$. Hint: If Y is an exponential random

variable with mean 1, and x is any non-negative number, show that $E[(Y \wedge x)^n] = n! \Gamma(n+1; x) + x^n e^{-x}$ and then use the surface interpretation of $E[(Y \wedge x)^n]$ and the fact that the n -th raw moment of Y is $n!$. Alternatively, you can just use brute force on the definition of $\Gamma(n+1; x)$, but that's not as much fun.

5. Individual loss amounts (ground up) for this year followed a Gamma distribution with mean 500 and variance 125,000. Next year you confidently expect individual loss amounts to inflate by 10% uniformly across the board. What will be the standard deviation next year for loss amounts that are subject to 250 deductible per loss with losses after deductible limited to 1,000 per loss? Please answer for the "per loss" variable, not the "per payment" variable.

Appendix A

An Inventory of Continuous Distributions

A.1 Introduction

The incomplete gamma function is given by

$$\Gamma(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t} dt, \quad \alpha > 0, x > 0$$

$$\text{with } \Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0.$$

When $\alpha \leq 0$ the integral does not exist. In that case, define

$$\Gamma(\alpha)G(\alpha; x) = \int_x^{\infty} t^{\alpha-1} e^{-t} dt, \quad x > 0.$$

Integration by parts produces the relationship

$$\Gamma(\alpha)G(\alpha; x) = -\frac{x^{\alpha}e^{-x}}{\alpha} + \frac{\Gamma(\alpha+1)}{\alpha}G(\alpha+1; x)$$

which allows for recursive calculation because for $\alpha > 0$, $\Gamma(\alpha)G(\alpha; x) = \Gamma(\alpha)[1 - \Gamma(\alpha; x)]$.

The incomplete beta function is given by

$$\beta(a, b; x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad a > 0, b > 0, 0 < x < 1.$$

A.2 Transformed beta family

A.2.3 Two-parameter distributions

A.2.3.1 Pareto— α, θ

$$\begin{aligned}
 f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} & F(x) &= 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha \\
 E[X^k] &= \frac{\theta^k \Gamma(k+1) \Gamma(\alpha-k)}{\Gamma(\alpha)}, & -1 < k < \alpha \\
 E[X^k] &= \frac{\theta^k k!}{(\alpha-1) \cdots (\alpha-k)}, & \text{if } k \text{ is an integer} \\
 E[X \wedge x] &= \frac{\theta}{\alpha-1} \left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1} \right], & \alpha \neq 1 \\
 E[X \wedge x] &= -\theta \log \left(\frac{\theta}{x+\theta}\right), & \alpha = 1 \\
 E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(k+1) \Gamma(\alpha-k)}{\Gamma(\alpha)} \beta[k+1, \alpha-k; x/(x+\theta)] + x^k \left(\frac{\theta}{x+\theta}\right)^\alpha, & \text{all } k \\
 \text{mode} &= 0
 \end{aligned}$$

A.2.3.2 Inverse Pareto— τ, θ

$$\begin{aligned}
 f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} & F(x) &= \left(\frac{x}{x+\theta}\right)^\tau \\
 E[X^k] &= \frac{\theta^k \Gamma(\tau+k) \Gamma(1-k)}{\Gamma(\tau)}, & -\tau < k < 1 \\
 E[X^k] &= \frac{\theta^k (-k)!}{(\tau-1) \cdots (\tau+k)}, & \text{if } k \text{ is a negative integer} \\
 E[(X \wedge x)^k] &= \theta^k \tau \int_0^{x/(x+\theta)} y^{\tau+k-1} (1-y)^{-k} dy + x^k \left[1 - \left(\frac{x}{x+\theta}\right)^\tau \right], & k > -\tau \\
 \text{mode} &= \theta \frac{\tau-1}{2}, & \tau > 1, \text{ else } 0
 \end{aligned}$$

A.2.3.3 Loglogistic— γ, θ

$$\begin{aligned}
 f(x) &= \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2} & F(x) &= u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
 E[X^k] &= \theta^k \Gamma(1+k/\gamma) \Gamma(1-k/\gamma), & -\gamma < k < \gamma \\
 E[(X \wedge x)^k] &= \theta^k \Gamma(1+k/\gamma) \Gamma(1-k/\gamma) \beta(1+k/\gamma, 1-k/\gamma; u) + x^k (1-u), & k > -\gamma \\
 \text{mode} &= \theta \left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, & \gamma > 1, \text{ else } 0
 \end{aligned}$$

A.2.3.4 Paralogistic— α, θ

This is a Burr distribution with $\gamma = \alpha$.

$$\begin{aligned}
 f(x) &= \frac{\alpha^2(x/\theta)^\alpha}{x[1+(x/\theta)^\alpha]^{\alpha+1}} & F(x) &= 1-u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\alpha} \\
 E[X^k] &= \frac{\theta^k \Gamma(1+k/\alpha) \Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)}, \quad -\alpha < k < \alpha^2 \\
 E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(1+k/\alpha) \Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)} \beta(1+k/\alpha, \alpha-k/\alpha; 1-u) + x^k u^\alpha, \quad k > -\alpha \\
 \text{mode} &= \theta \left(\frac{\alpha-1}{\alpha^2+1} \right)^{1/\alpha}, \quad \alpha > 1, \text{ else } 0
 \end{aligned}$$

A.2.3.5 Inverse paralogistic— τ, θ

This is an inverse Burr distribution with $\gamma = \tau$.

$$\begin{aligned}
 f(x) &= \frac{\tau^2(x/\theta)^{\tau^2}}{x[1+(x/\theta)^\tau]^{\tau+1}} & F(x) &= u^\tau, \quad u = \frac{(x/\theta)^\tau}{1+(x/\theta)^\tau} \\
 E[X^k] &= \frac{\theta^k \Gamma(\tau+k/\tau) \Gamma(1-k/\tau)}{\Gamma(\tau)}, \quad -\tau^2 < k < \tau \\
 E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau+k/\tau) \Gamma(1-k/\tau)}{\Gamma(\tau)} \beta(\tau+k/\tau, 1-k/\tau; u) + x^k [1-u^\tau], \quad k > -\tau^2 \\
 \text{mode} &= \theta(\tau-1)^{1/\tau}, \quad \tau > 1, \text{ else } 0
 \end{aligned}$$

A.3 Transformed gamma family

A.3.2 Two-parameter distributions

A.3.2.1 Gamma— α, θ

$$\begin{aligned}
 f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x \Gamma(\alpha)} & F(x) &= \Gamma(\alpha; x/\theta) \\
 M(t) &= (1-\theta t)^{-\alpha} & E[X^k] &= \frac{\theta^k \Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad k > -\alpha \\
 E[X^k] &= \theta^k (\alpha+k-1) \cdots \alpha, \quad \text{if } k \text{ is an integer}
 \end{aligned}$$

$$\begin{aligned}
 E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha+k)}{\Gamma(\alpha)} \Gamma(\alpha+k; x/\theta) + x^k [1-\Gamma(\alpha; x/\theta)], \quad k > -\alpha \\
 &= \alpha(\alpha+1) \cdots (\alpha+k-1) \theta^k \Gamma(\alpha+k; x/\theta) + x^k [1-\Gamma(\alpha; x/\theta)], \quad k \text{ an integer} \\
 \text{mode} &= \theta(\alpha-1), \quad \alpha > 1, \text{ else } 0
 \end{aligned}$$

A.3.2.2 Inverse gamma— α, θ

$$\begin{aligned}
 f(x) &= \frac{(\theta/x)^\alpha e^{-\theta/x}}{x\Gamma(\alpha)} & F(x) &= 1 - \Gamma(\alpha; \theta/x) \\
 E[X^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}, \quad k < \alpha & E[X^k] &= \frac{\theta^k}{(\alpha - 1) \cdots (\alpha - k)}, \quad \text{if } k \text{ is an integer} \\
 E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)} [1 - \Gamma(\alpha - k; \theta/x)] + x^k \Gamma(\alpha; \theta/x) \\
 &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)} G(\alpha - k; \theta/x) + x^k \Gamma(\alpha; \theta/x), \quad \text{all } k \\
 &= \frac{\theta^k}{(\alpha - 1) \cdots (\alpha - k)} G(\alpha - k; \theta/x) + x^k \Gamma(\alpha; \theta/x), \quad k \text{ is an integer} \\
 \text{mode} &= \theta/(\alpha + 1)
 \end{aligned}$$

A.3.2.3 Weibull— θ, τ

$$\begin{aligned}
 f(x) &= \frac{\tau(x/\theta)^\tau e^{-(x/\theta)^\tau}}{x} & F(x) &= 1 - e^{-(x/\theta)^\tau} \\
 E[X^k] &= \theta^k \Gamma(1 + k/\tau), \quad k > -\tau \\
 E[(X \wedge x)^k] &= \theta^k \Gamma(1 + k/\tau) \Gamma[1 + k/\tau; (x/\theta)^\tau] + x^k e^{-(x/\theta)^\tau}, \quad k > -\tau \\
 \text{mode} &= \theta \left(\frac{\tau - 1}{\tau} \right)^{1/\tau}, \quad \tau > 1, \text{ else } 0
 \end{aligned}$$

A.3.2.4 Inverse Weibull— θ, τ

$$\begin{aligned}
 f(x) &= \frac{\tau(\theta/x)^\tau e^{-(\theta/x)^\tau}}{x} & F(x) &= e^{-(\theta/x)^\tau} \\
 E[X^k] &= \theta^k \Gamma(1 - k/\tau), \quad k < \tau \\
 E[(X \wedge x)^k] &= \theta^k \Gamma(1 - k/\tau) \{1 - \Gamma[1 - k/\tau; (\theta/x)^\tau]\} + x^k [1 - e^{-(\theta/x)^\tau}], \quad \text{all } k \\
 &= \theta^k \Gamma(1 - k/\tau) G[1 - k/\tau; (\theta/x)^\tau] + x^k [1 - e^{-(\theta/x)^\tau}] \\
 \text{mode} &= \theta \left(\frac{\tau}{\tau + 1} \right)^{1/\tau}
 \end{aligned}$$

A.3.3 One-parameter distributions

A.3.3.1 Exponential— θ

$$\begin{aligned} f(x) &= \frac{e^{-x/\theta}}{\theta} & F(x) &= 1 - e^{-x/\theta} \\ M(t) &= (1 - \theta t)^{-1} & E[X^k] &= \theta^k \Gamma(k+1), \quad k > -1 \\ E[X^k] &= \theta^k k!, \quad \text{if } k \text{ is an integer} \\ E[X \wedge x] &= \theta(1 - e^{-x/\theta}) \\ E[(X \wedge x)^k] &= \theta^k \Gamma(k+1) \Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k > -1 \\ &= \theta^k k! \Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k \text{ an integer} \\ \text{mode} &= 0 \end{aligned}$$

A.3.3.2 Inverse exponential— θ

$$\begin{aligned} f(x) &= \frac{\theta e^{-\theta/x}}{x^2} & F(x) &= e^{-\theta/x} \\ E[X^k] &= \theta^k \Gamma(1-k), \quad k < 1 \\ E[(X \wedge x)^k] &= \theta^k \Gamma(1-k) G(1-k; \theta/x) + x^k (1 - e^{-\theta/x}), \quad \text{all } k \\ \text{mode} &= \theta/2 \end{aligned}$$

A.4 Other distributions

A.4.1.1 Lognormal— μ, σ (μ can be negative)

$$\begin{aligned} f(x) &= \frac{1}{x\sigma\sqrt{2\pi}} \exp(-z^2/2) = \phi(z)/(\sigma x), \quad z = \frac{\log x - \mu}{\sigma} & F(x) &= \Phi(z) \\ E[X^k] &= \exp(k\mu + k^2\sigma^2/2) \\ E[(X \wedge x)^k] &= \exp(k\mu + k^2\sigma^2/2) \Phi\left(\frac{\log x - \mu - k\sigma^2}{\sigma}\right) + x^k [1 - F(x)] \\ \text{mode} &= \exp(\mu - \sigma^2) \end{aligned}$$

A.4.1.2 Inverse Gaussian— μ, θ

$$\begin{aligned} f(x) &= \left(\frac{\theta}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\theta z^2}{2x}\right\}, \quad z = \frac{x - \mu}{\mu} \\ F(x) &= \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] + \exp(2\theta/\mu) \Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \quad y = \frac{x + \mu}{\mu} \\ M(t) &= \exp\left[\frac{\theta}{\mu} \left(1 - \sqrt{1 - \frac{2t\mu^2}{\theta}}\right)\right] & E[X] &= \mu, \quad \text{Var}[X] = \mu^3/\theta \\ E[X \wedge x] &= x - \mu z \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] - \mu y \exp(2\theta/\mu) \Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right] \end{aligned}$$

A.4.1.3 Single-parameter Pareto— α, θ

$$\begin{aligned}
 f(x) &= \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, & x > \theta & & F(x) &= 1 - (\theta/x)^\alpha, & x > \theta \\
 E[X^k] &= \frac{\alpha\theta^k}{\alpha - k}, & k < \alpha & & E[(X \wedge x)^k] &= \frac{\alpha\theta^k}{\alpha - k} - \frac{k\theta^\alpha}{(\alpha - k)x^{\alpha-k}} \\
 \text{mode} &= \theta
 \end{aligned}$$

Note: Although there appears to be two parameters, only α is a true parameter. The value of θ must be set in advance.

A.5 Distributions with finite support

For these two distributions, the scale parameter θ is assumed known.

A.5.1.1 Generalized beta— a, b, θ, τ

$$\begin{aligned}
 f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{\tau}{x}, & 0 < x < \theta, & & u &= (x/\theta)^\tau \\
 F(x) &= \beta(a, b; u) \\
 E[X^k] &= \frac{\theta^k \Gamma(a+b) \Gamma(a+k/\tau)}{\Gamma(a) \Gamma(a+b+k/\tau)}, & k > -a\tau \\
 E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(a+b) \Gamma(a+k/\tau)}{\Gamma(a) \Gamma(a+b+k/\tau)} \beta(a+k/\tau, b; u) + x^k [1 - \beta(a, b; u)]
 \end{aligned}$$

A.5.1.2 beta— a, b, θ

$$\begin{aligned}
 f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{1}{x}, & 0 < x < \theta, & & u &= x/\theta \\
 F(x) &= \beta(a, b; u) \\
 E[X^k] &= \frac{\theta^k \Gamma(a+b) \Gamma(a+k)}{\Gamma(a) \Gamma(a+b+k)}, & k > -a \\
 E[X^k] &= \frac{\theta^k a(a+1) \cdots (a+k-1)}{(a+b)(a+b+1) \cdots (a+b+k-1)}, & \text{if } k \text{ is an integer} \\
 E[(X \wedge x)^k] &= \frac{\theta^k a(a+1) \cdots (a+k-1)}{(a+b)(a+b+1) \cdots (a+b+k-1)} \beta(a+k, b; u) \\
 &+ x^k [1 - \beta(a, b; u)]
 \end{aligned}$$

Appendix B

An Inventory of Discrete Distributions

B.2 The $(a, b, 0)$ class

B.2.1.1 Poisson— λ

$$\begin{aligned} p_0 &= e^{-\lambda}, & a &= 0, & b &= \lambda & p_k &= \frac{e^{-\lambda} \lambda^k}{k!} \\ E[N] &= \lambda, & \text{Var}[N] &= \lambda & P(z) &= e^{\lambda(z-1)} \end{aligned}$$

B.2.1.2 Geometric— β

$$\begin{aligned} p_0 &= 1/(1+\beta), & a &= \beta/(1+\beta), & b &= 0 & p_k &= \frac{\beta^k}{(1+\beta)^{k+1}} \\ E[N] &= \beta, & \text{Var}[N] &= \beta(1+\beta) & P(z) &= [1-\beta(z-1)]^{-1} \end{aligned}$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 Binomial— $q, m, (0 < q < 1, m \text{ an integer})$

$$\begin{aligned} p_0 &= (1-q)^m, & a &= -q/(1-q), & b &= (m+1)q/(1-q) \\ p_k &= \binom{m}{k} q^k (1-q)^{m-k}, & k &= 0, 1, \dots, m \\ E[N] &= mq, & \text{Var}[N] &= mq(1-q) & P(z) &= [1+q(z-1)]^m \end{aligned}$$

B.2.1.4 Negative binomial— β, r

$$\begin{aligned} p_0 &= (1+\beta)^{-r}, & a &= \beta/(1+\beta), & b &= (r-1)\beta/(1+\beta) \\ p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \\ E[N] &= r\beta, & \text{Var}[N] &= r\beta(1+\beta) & P(z) &= [1-\beta(z-1)]^{-r} \end{aligned}$$