**Measure and Integration Preliminary Exam, January 2014**

Solve four problems. Indicate which four.

1. Let $h$ be a bounded measurable function on $\mathbb{R}$ such that
   \[ \lim_{x \to \pm \infty} \frac{1}{x} \int_0^x h(t) dt = 0. \]
   Show that for all functions $F \in L^1(\mathbb{R})$,
   \[ \lim_{\lambda \to \infty} \int \mathbb{R} F(t) h(\lambda t) dt = 0. \]
   Does $h(t) = \cos t$ satisfy the hypothesis, hence the conclusion? Hint: Start with indicators of intervals.

2. Prove that if $f : [0, \infty) \mapsto [0, \infty)$ is non-increasing and Lebesgue integrable, then $\lim_{x \to \infty} xf(x) = 0$. Give an example of a continuous Lebesgue integrable function on $[0, \infty)$ for which $\limsup_{x \to \infty} xf(x) = \infty$.

3. Let $(S, \mathcal{S}, \mu)$ be a measure space.
   
   (a) Prove that, if $\mu(S) < \infty$, and each $f_n, n \in \mathbb{N}$, is measurable, then
   \[ f_n \to 0 \text{ in measure} \iff \int_S \frac{|f_n|}{1 + |f_n|} d\mu \to 0. \]
   
   (b) Prove or disprove (e.g. by giving a counterexample) each of the two implications if $\mu(S) = \infty$.

4. (a) Use a real analysis theorem to show that $\sum_k \sum_{\ell} a_{k\ell} = \sum_{\ell} \sum_k a_{k\ell}$ for $a_{k\ell} \geq 0$. Then show that if $\mu_k, k \in \mathbb{N}$, are measures on $(S, \mathcal{S})$, so is the set function $\mu$ given by $\mu(A) = \sum_{k=1}^{\infty} \mu_k(A), A \in \mathcal{S}$.
   
   (b) Let $(S, \mathcal{S}) = (\mathbb{R}, \mathcal{B})$, assume $\sum_k \mu_k[-n, n]$ is finite for all $n$, and let $\mu_k = \lambda_k + \nu_k$ be the Lebesgue decomposition of $\mu_k$ for each $k$ (with Lebesgue measure $m$ being a mutually singular, and $\nu_k$ is absolutely continuous w.r.t. $m$). Prove that if $\lambda = \sum_k \lambda_k$ and $\nu = \sum_k \nu_k$, then $\mu = \lambda + \nu$ is the Lebesgue decomposition of $\mu$.
   
   (c) Show that if $F_k : [a, b] \mapsto [0, \infty)$ are non-decreasing, right continuous and non-negative functions, and if $F(x) := \sum_k F_k(x) < \infty$ for all $x$ in $[a, b]$, then $F$ is also right continuous (and, obviously, non-decreasing and non-negative) and
   \[ F'(x) = \sum_k F'_k(x) \]
   for almost all $x$ in $[a, b]$.

5. (a) Prove the generalized Minkowski inequality, that is, prove that if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are sigma-finite measure spaces and $f : X \times Y \mapsto \mathbb{R}$ is $\mathcal{A} \otimes \mathcal{B}$-measurable, then
   \[ \left\| \|f\|_{L^p(X,\mu)} \right\|_{L^q(Y,\nu)} \leq \left\| \|f\|_{L^p(Y,\nu)} \right\|_{L^1(X,\mu)} \]
   for all $p \geq 1$. Hint: duality of $L_p$ spaces and a famous inequality may help.
   
   (b) Let $\| \cdot \|_p$ stand for the $L_p(\mathbb{R})$-norm with respect to the Lebesgue measure. Show that if $p > 1$ and $f \in L_p(\mathbb{R})$, then the ‘mean functional’ of $f$,
   \[ F(x) := \frac{1}{x} \int_0^x f(y) dy = \int_0^1 f(xt) dt, \]
   is also in $L_p(\mathbb{R})$ and, moreover,
   \[ \|F\|_p \leq q \|f\|_p \]
   where $q$ is conjugate of $p$, that is $p^{-1} + q^{-1} = 1$. 