Measure and Integration Prelim, January 2011

1. Let \( f : [0, 1] \to \mathbb{R} \) be bounded.
   (a) Show that the set where \( f \) is continuous is Lebesgue measurable (even if \( f \) is not Lebesgue measurable).
   (b) Show that if \( f \) is not continuous on a set of full Lebesgue measure, then \( f \) is not Riemann integrable.
   
   Hint: consider the standard partition of \([0, 1]\) into \(2^n\) subintervals, and define \( F_n(x) \) to be the sup of \( f \) over the interval containing \( x \) and define \( f_n(x) \) to be the inf of \( f \) over this interval.

2. Let \((X, \mathcal{F}, \mu)\) be a measure space. Suppose that \( f \) is a measurable nonnegative function satisfying \( \int f \, d\mu = 1 \). Compute \( \lim_{n \to \infty} \int n \log \left( 1 + \left( \frac{f(x)}{n} \right)^\alpha \right) \, d\mu(x) \) in three different cases:
   (a) \( 0 < \alpha < 1 \)
   (b) \( \alpha = 1 \)
   (c) \( \alpha > 1 \)
   
   Justify your answer in each case.
   
   Hint: writing \( n = n^\alpha n^{1-\alpha} \), and the inequalities \( \log(1 + u) \leq u \) and \( 1 + u^\alpha \leq (1 + u)^\alpha \) for \( u \geq 0, \alpha \geq 1 \) may be useful.

3. (a) Suppose \( p, q \in (1, \infty) \) satisfy \( 1/p + 1/q = 1 \), and \( a, b \in (0, \infty) \). Prove that \( ab \leq a^{p}/p + b^{q}/q \). Hint: it may help to write the inequality in terms of \( s = p \log a \) and \( t = q \log b \).
   (b) State and prove Hölder’s inequality for \( p, q \in (1, \infty) \). Hint: first show that it is sufficient to prove the case where \( \|f\|_p = \|g\|_q = 1 \), then use (a).

4. Let \( f(x, y) \in L^1(Q) \) where \( Q = [0, 1] \times [0, 1] \) is the unit square in \( \mathbb{R}^2 \). Suppose that for any continuous function \( g(y) \) on \([0, 1]\) we know
   \[
   \int f(x, y)g(y) \, dy = 0 \quad \text{for almost every } x \in [0, 1].
   \]
   
   Prove that \( f = 0 \) a.e. on \( Q \).