MEASURE AND INTEGRATION (MATH 5111)
QUALIFYING EXAM - JANUARY 2010

Below $(X,\mathcal{F},\mu)$ denotes a general measure space, and $(\mathbb{R},\mathcal{L},dx)$ denotes the real-line equipped with the Lebesgue σ-algebra and the Lebesgue measure.

1. Recall that a sequence $(f_n : n\in\mathbb{N})$ of real-valued $\mathcal{F}$-measurable functions on X converges μ-almost uniformly to a function f if for every $\epsilon>0$ there exists $E\in\mathcal{F}$ such that $\mu(E)<\epsilon$ and on $X\setminus E$, $f_n \underset{n\to\infty}{\to} f$ uniformly.
   - (a) Suppose that $f_n \underset{n\to\infty}{\to} f$, μ-almost uniformly. Prove that $f_n \underset{n\to\infty}{\to} f$ in μ-measure and μ-almost surely.
   - (b) (Egoroff's Theorem) Assume that $\mu(X)<\infty$. Prove that if $f_n \underset{n\to\infty}{\to} f$, μ-almost surely, then $f_n \underset{n\to\infty}{\to} f$, μ-almost uniformly.
   - (c) Show by example that the condition $\mu(X)<\infty$ in part (b) could not be relaxed.
2. Calculate
$$\lim_{n\to\infty}\int_0^\infty \left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)dx.$$
3. For $f:\mathbb{R}\to\mathbb{R}$ and $t\in\mathbb{R}$ define a function T_tf by letting $(T_tf)(x)=f(x-t)$.
   - (a) Let $p\in[1,\infty)$. Prove that for $f\in L^p(dx)$, $\|T_tf-f\|_p \underset{t\to 0}{\to} 0$.
   - (b) Show by example that the claim above fails for $p=\infty$.
4. Let $p\in(1,\infty)$ and let $q\in(1,\infty]$ denote the conjugate exponent, $\frac{1}{p}+\frac{1}{q}=1$. Suppose that $f:\mathbb{R}\times\mathbb{R}\to[0,\infty]$ is a nonnegative $\mathcal{L}\times\mathcal{L}$-measurable function.
   - (a) (Generalized Minkowski's inequality) Let $g:\mathbb{R}\to[0,\infty]$ be a nonnegative $\mathcal{L}$-measurable function. Prove:
$$\iint f(x,y)g(x)dxdy \le \|g\|_q \int\left(\int f(x,y)^p dx\right)^{1/p} dy.$$
   - (b) Conclude from (a) that
$$\left(\int\left(\int f(x,y)dy\right)^p dx\right)^{1/p} \le \int\left(\int f(x,y)^p dx\right)^{1/p} dy.$$
   - (c) (Hardy's inequality) Let $h\in L^p(dx)$. Define a function Th by letting
$$(Th)(y)=\begin{cases}\frac{1}{y}\int_0^y h(x)dx & y>0\\ 0 & \text{otherwise.}\end{cases}$$

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Prove

$$\|Th\|_p \leq \frac{p}{p-1}\|h\|_p.$$

Hint: $(Th)(y) = \int_{[0,\infty)} h(x)\mathbf{1}_{[0,y]}(x)\frac{dx}{y} = \int_{[0,1]} h(yu)du$.

5. Determine whether each of the following statements is TRUE or FALSE and justify your answer (prove it or find a counterexample to the statement).

   (a) Suppose that f is a function on $[0,1]$ which is differentiable at each point $x \in [0,1]$. Then f is absolutely continuous on $[0,1]$.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is a function of bounded variation and $f'(x) = 0$, dx-almost surely, then there exists $C \in \mathbb{R}$ such that $f(x) = C$, dx-almost surely.

   (c) Let f be a function in $L^1_{loc}(dx)$ (i.e. integrable over bounded intervals). Assume that $|\int_{[k/2^j,(k+1)/2^j)} f(x)\sin x dx| \leq 4^{-j}$ for all $k \in \mathbb{Z}, j \in \mathbb{N}$. Then $f = 0$, dx-almost surely.

   (d) If $(X, \mathcal{F}, \mu)$ is a complete measure space, then so is the product space $(X \times X, \mathcal{F} \times \mathcal{F}, \mu \times \mu)$. (Recall that a measure space is complete if all subsets of sets of measure 0 are measurable).

   (e) If $f : [0,1] \to \mathbb{R} \cup \{-\infty, +\infty\}$ is Riemann integrable on $[\epsilon, 1]$ for every $\epsilon \in (0,1)$ and the improper Riemann integral $\lim_{\epsilon \searrow 0} \int_\epsilon^1 f(x)dx$ exists (and is finite) then $f \in L^1([0,1], dx)$.