Below \((X, \mathcal{F}, \mu)\) denotes a general measure space, and \((\mathbb{R}, \mathcal{L}, dx)\) denotes the real-line equipped with the Lebesgue \(\sigma\)-algebra and the Lebesgue measure.

1. Recall that a sequence \(\{f_n : n \in \mathbb{N}\}\) of real-valued \(\mathcal{F}\)-measurable functions on \(X\) converges \(\mu\)-almost uniformly to a function \(f\) if for every \(\epsilon > 0\) there exists \(E \in \mathcal{F}\) such that \(\mu(E) < \epsilon\) and on \(X \setminus E\), \(f_n \to f\) uniformly.

(a) Suppose that \(f_n \to f\), \(\mu\)-almost uniformly. Prove that \(f_n \to f\) in \(\mu\)-measure and \(\mu\)-almost surely.

(b) (Egoroff’s Theorem) Assume that \(\mu(X) < \infty\). Prove that if \(f_n \to f\), \(\mu\)-almost surely, then \(f_n \to f\), \(\mu\)-almost uniformly.

(c) Show by example that the condition \(\mu(X) < \infty\) in part (b) could not be relaxed.

2. Calculate
\[
\lim_{n \to \infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin \left(\frac{x}{n}\right) \, dx.
\]

3. For \(f : \mathbb{R} \to \mathbb{R}\) and \(t \in \mathbb{R}\) define a function \(T_t f\) by letting \((T_t f)(x) = f(x - t)\).

(a) Let \(p \in [1, \infty)\). Prove that for \(f \in L^p(dx)\), \(\|T_t f - f\|_p \to 0\) as \(t \to 0\).

(b) Show by example that the claim above fails for \(p = \infty\).

4. Let \(p \in (1, \infty)\) and let \(q \in (1, \infty]\) denote the conjugate exponent, \(\frac{1}{p} + \frac{1}{q} = 1\). Suppose that \(f : \mathbb{R} \times \mathbb{R} \to [0, \infty]\) is a nonnegative \(\mathcal{L} \times \mathcal{L}\)-measurable function.

(a) (Generalized Minkowski’s inequality) Let \(g : \mathbb{R} \to [0, \infty]\) be a nonnegative \(\mathcal{L}\)-measurable function. Prove:
\[
\iint f(x, y)g(x) \, dx \, dy \leq \|g\|_q \int \left(\int f(x, y)^p \, dx\right)^{1/p} \, dy.
\]

(b) Conclude from (a) that
\[
\left(\int \left(\int f(x, y)^p \, dy\right)^{1/p} \, dx\right)^{1/p} \leq \int \left(\int f(x, y)^p \, dx\right)^{1/p} \, dy.
\]

(c) (Hardy’s inequality) Let \(h \in L^p(dx)\). Define a function \(Th\) by letting
\[
(Th)(y) = \begin{cases} \frac{1}{y} \int_0^y h(x) \, dx & \text{if } y > 0 \\ 0 & \text{otherwise}. \end{cases}
\]

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Prove
\[ \|Th\|_p \leq \frac{p}{p-1}\|h\|_p. \]

Hint: 
\[ (Th)(y) = \int_{[0,\infty)} h(x) \mathbf{1}_{[0,y]}(x) \frac{dx}{y} = \int_{[0,1]} h(yu) du. \]

5. Determine whether each of the following statements is TRUE or FALSE and justify your answer (prove it or find a counterexample to the statement).

(a) Suppose that \( f \) is a function on \([0, 1]\) which is differentiable at each point \( x \in [0, 1] \). Then \( f \) is absolutely continuous on \([0, 1]\).

(b) If \( f : \mathbb{R} \to \mathbb{R} \) is a function of bounded variation and \( f'(x) = 0 \), \( dx \)-almost surely, then there exists \( C \in \mathbb{R} \) such that \( f(x) = C \), \( dx \)-almost surely.

(c) Let \( f \) be a function in \( L_{loc}^1(dx) \) (i.e. integrable over bounded intervals). Assume that \( | \int_{[k/2^j,(k+1)/2^j]} f(x) \sin x \, dx | \leq 4^{-j} \) for all \( k \in \mathbb{Z}, j \in \mathbb{N} \). Then \( f = 0 \), \( dx \)-almost surely.

(d) If \((X, \mathcal{F}, \mu)\) is a complete measure space, then so is the product space \((X \times X, \mathcal{F} \times \mathcal{F}, \mu \times \mu)\). (Recall that a measure space is complete if all subsets of sets of measure 0 are measurable).

(e) If \( f : [0, 1] \to \mathbb{R} \cup \{-\infty, +\infty\} \) is Riemann integrable on \([\epsilon, 1]\) for every \( \epsilon \in (0, 1) \) and the improper Riemann integral \( \lim_{\epsilon \searrow 0} \int_{\epsilon}^1 f(x) \, dx \) exists (and is finite) then \( f \in L^1([0, 1], dx) \).