Real Analysis Prelim, August 2014

Notation: $\mathcal{L}$ is the Lebesgue σ-algebra of subsets of $\mathbb{R}$, m is the Lebesgue measure on $\mathcal{L}$, and the term "Borel measurable" refers to the Borel σ-algebra on $\mathbb{R}$.

1. For every $a, b \in \mathbb{R}$ let $L_{a,b} : \mathbb{R} \to \mathbb{R}$ be the function $L_{a,b}(x) = ax + b$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded from below and define a set $A \subset \mathbb{R}^2$ and a function $\bar{f} : \mathbb{R} \to \mathbb{R}$ by
$$A = \{(a, b) \in \mathbb{R}^2 : L_{a,b}(z) \leq f(z) \text{ for all } z \in \mathbb{R}\}, \qquad \bar{f}(x) = \sup_{(a,b)\in A} L_{a,b}(x).$$
Note that the function $\overline{f}$ is called the convex minorant of f, the largest convex function less than or equal to f. Prove that $\bar{f}$ is Borel measurable.

2. (a) Prove that $\lim_{n\to\infty} \sum_{j=0}^{n} x^{j-j^2/2n} = \dfrac{1}{1-x}$ for $0 < x < 1$.
Hint: fix x and use the dominated convergence theorem for the series as an integral over non-negative integers with respect to the counting measure (integrate functions that have j as the variable).
   (b) Compute $\lim_{n\to\infty} \int_{[0,1]} \sqrt{\sum_{j=0}^{n} x^{j-j^2/2n}}\, dm(x)$.
Hint: again use the dominated convergence theorem, but this time for the integral over the unit interval with respect to m.

3. Let μ be a measure on $\mathcal{L}$, defined by $\mu(A) = \int_{A\cap(0,\infty)} \frac{1}{y} dm(y)$. Suppose that $f, g \in L^1(\mu)$. Let $H(x) = \int_0^\infty f(x/y)g(y)d\mu(y)$. Prove the following:
   (a) The integral above defining H exists μ-a.e., H is measurable, and $H \in L^1(\mu)$
   (b) $H(x) = \int_0^\infty g(x/y)f(y)d\mu(y)$.

4. Let $p_1, p_2 \in (1, \infty)$.
   (a) Find $r = r(p_1, p_2) \in (0, \infty)$ such that $(fg)^r \in L^1(m)$ whenever $f \in L^{p_1}(m)$ and $g \in L^{p_2}(m)$.
   (b) Show that if $r' \neq r(p_1, p_2)$, then there exist nonnegative $f \in L^{p_1}(m)$ and $g \in L^{p_2}(m)$ such that $(fg)^{r'} \notin L^1(m)$.

5. (a) Define: $f{:}[0, 1]{\to}\mathbb{R}$ is absolutely continuous.
   (b) Prove that $f{:}[0, 1]{\to}\mathbb{R}$ is absolutely continuous if and only if it is continuous, bounded variation, and whenever $A \in \mathcal{L}$ and $m(A) = 0$, then $f(A) \in \mathcal{L}$ and $m(f(A)) = 0$. Here $f(A)$ is the image of the set A under the function f.