Measure and Integration Prelim, August 2013

Below m denotes the Lebesgue measure on $\mathbb{R}$.

1. (a) State the dominated convergence theorem (DCT)

   (b) Recall the bounded convergence theorem (BCT)

   > Suppose that μ is a **finite** measure. Let $f, (f_n : n \in \mathbb{N})$ be measurable functions satisfying that $\sup_n \|f_n\|_\infty < \infty$ and $\lim_{n\to\infty} f_n = f$ μ-a.e. Then $\int f_n d\mu \to \int f d\mu$.

   Clearly, DCT $\Rightarrow$ BCT. Prove the converse.
   (Hint: A fast way to solve this problem, but maybe not the only way, is by change of measure.)

2. Suppose that $w \in L^1(m)$ has the property that $\int w\varphi dm = 0$ for all $\varphi \in L^\infty(m) \cap L^1(m)$ satisfying $\int \varphi dm = 0$. Show that $w \equiv 0$, m-a.e.

3. Let F be of bounded variation, and let dF denote the corresponding signed measure. Prove that if φ is a continuous function with compact support and continuous derivative, then

$$\int F\varphi' dm = -\int \varphi dF.$$

4. Show that on $\mathbb{R}$, if f is continuous at $x = 1$ and $g \in L^1(m)$, then for every $\alpha \in (-\infty, 1)$,

$$\int_{-n^\alpha}^{n^\alpha} f(1 + x/n)g(x)dm(x)$$

   converges as $n \to \infty$. Show by example that this integral may not converge for $\alpha = 1$.

5. Let $q_1, q_2, \ldots$ be an enumeration of rationals in $[0, 1]$. Consider the infinite series

$$s(x) = \sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{|x - q_n|}}.$$

   (a) Prove that s converges m-a.e.

   (b) Prove that s is unbounded on any non-empty open subinterval of $[0, 1]$.