2012 Real Analysis Prelim Exam

Justify your reasoning in all problems.

(1) (a) Does there exist a measure \( \mu \) on the set of rational numbers \( \mathbb{Q} \) such that all intervals are measurable and \( \mu([0,q)) = q \) for any positive rational number \( q \)? Prove or disprove.

(b) Suppose \((\mathbb{R}, \mathcal{A}, \mu)\) is the measure space where \( \mathcal{A} \) is the \( \sigma \)-algebra of all subsets and \( \mu \) is the counting measure (which means that \( \mu(A) = |A| \), the cardinality of \( A \), if \( A \) is a finite set, and \( \mu(A) = \infty \) if \( A \) is an infinite set). Prove or disprove that the function
\[
F(x) = \begin{cases} 
  e^{-|x|} & \text{if } x \in \mathbb{Q} \\
  0 & \text{if } x \notin \mathbb{Q}
\end{cases}
\]
is integrable.

(c) in the same situation as in (b), prove or disprove that finitely supported functions are dense in \( L^1(\mathbb{R}, \mathcal{A}, \mu) \).

(2) Suppose \((\mathbb{R}, \mathcal{A}, \mu)\) is a measure space. Prove that
\[
\int_X f_n \, d\mu \to \int_X f \, d\mu
\]
if \( f_n, g_n, f, g \) are integrable, \( f_n \to f \) and \( g_n \to g \) \( \mu \)-a.e., \( |f_n| \leq g_n \) for all \( n \), and \( \int_X g_n \, d\mu \to \int_X g \, d\mu \).

Hint: use Fatou’s lemma for \( g_n + f_n \) and \( g_n - f_n \), or for \( 2g_n - |f_n - f| \).

(3) Prove that \( F(x) = \sum_{n=0}^{\infty} e^{-n} \cos(1 + n^2 x^2) \) is a differentiable function on \( \mathbb{R} \).

(4) (a) Suppose \((\mathbb{R}, \mathcal{A}, \mu)\) is a measurable space where \( \mathcal{A} \) is the Borel \( \sigma \)-algebra of subsets and \( \mu \) is a Lebesgue-Stieltjes measure which is translation invariant in the sense that \( \mu(A) = \mu(\{x : x + y \in A\}) \) for any Borel set \( A \) and any \( y \in \mathbb{R} \). Prove that if \( \mu \) is finite on bounded intervals, then it is a multiple of the Lebesgue measure.

(b) Does there exist a non-zero \( f \in L^1([0, \infty), \mathcal{A}, \lambda) \) such that \( \int_{[0,q)} f \, d\lambda = 0 \) for any positive rational number \( q \)? Here \(([0, \infty), \mathcal{A}, \lambda)\) is the measure space with the Borel \( \sigma \)-algebra \( \mathcal{A} \) and the Lebesgue measure \( \lambda \). Prove or disprove.

(5) State and prove the Minkowski inequality.