1. In this problem, prove assertions a) and b) by invoking limit theorems for integrals. Check carefully the conditions that allow you to apply these theorems.

a) \( \lim_{n \to \infty} \int_{0}^{1} \frac{n + n^k x^k}{(1 + \sqrt{x})^n} \, dx = 0 \) for all \( k \in \mathbb{N} \).

b) If \( f : [0, \infty) \to \mathbb{R} \) is a non-negative, monotone non-increasing function which is Lebesgue integrable on \([0, \infty)\), then \( \int_{0}^{\infty} f(x) \, dx = \lim_{n \to \infty} \int_{0}^{x_n} (f(x) - f(x_n)) \, dx \) whenever \( x_n \nearrow \infty \) and, as a consequence, \( \lim_{x \to \infty} x f(x) = 0 \).

c) Is \( \lim_{x \to \infty} x f(x) = 0 \) true if in b) we replace the monotonicity condition by uniform continuity? Justify your answer.

2. In this question, \( f_n, f \) are measurable functions defined on a general measure space \((X, \Sigma, \mu)\). Answer whether each of the following statements is true or false, provide a proof if your answer is ‘true’, and provide a counterexample if your answer is ‘false’.

a) If \( \sum_{n=1}^{\infty} \int |f_n - f|^p \, d\mu < \infty \) for some \( p > 0 \) then \( f_n \to f \) \( \mu \)-a.e.

b) \( f_n \to f \) in \( L^p(\mu) \) (for some \( p \geq 1 \)) implies \( f_n \to f \) \( \mu \)-a.e.

3. Let \( f : [0, 1] \to \mathbb{R} \) satisfy the property that there exists \( M < \infty \) such that \( \|f\|_p \leq M \) for all \( 1 \leq p < \infty \), where \( \|\cdot\|_p \) denotes the \( L^p \) norm for Lebesgue measure on \([0, 1]\).

a) Does it follow that \( f \in L^\infty([0, 1]) \)?

b) What if \( f \in L^p([0, 1]) \) but no such constant \( M \) exists? (That is, are there any functions that are in \( L^p([0, 1]) \) for all \( 1 \leq p < \infty \) but not in \( L^\infty([0, 1]) \)?)

4. Let \( F \) and \( G \) be two functions of bounded variation on \([a, b], -\infty < a < b < +\infty \). Assume \( F \) is continuous and \( G \) is right continuous, and let \( \mu_F \) and \( \mu_G \) be the corresponding Lebesgue-Stieltjes measures on \([a, b]\), that is, \( \mu_F \) is the only Borel measure on \([a, b] \) such that \( \mu_F(a, x] = F(x) - F(a) \), \( a < x \leq b \), and \( \mu_G \) is defined analogously. Prove the integration by parts formula

\[
\int_{(a, b]} F(x) \, d\mu_G(x) = F(b)G(b) - F(a)G(a) - \int_{(a, b]} G(x) \, d\mu_F(x).
\]

Hint: Reduction to the case of \( \mu_F \) and \( \mu_G \) positive (and \( \mu_F \) atomless), which does require using an important theorem, will allow you to apply a theorem on product measures. Check carefully and explicitly the hypotheses of any theorems you invoke.