Measure and Integration Aug 2010
Qualifying Exam

- Answer all problems.
- State clearly what results you rely on.
- Notation: $(X, \mathcal{F}, \mu)$ denotes a measure space; m denotes the Lebesgue measure on $\mathbb{R}$.

1. A sequence $(f_n : n \in \mathbb{N})$ of functions in $L^1(\mu)$ is called *Uniformly Integrable (UI)* if

$$\lim_{\epsilon \searrow 0} \sup_{\{A \in \mathcal{F} : \mu(A) < \epsilon\}} \int_A |f_n| d\mu = 0.$$

Assume that $\mu(X) < \infty$, and let $(f_n : n \in \mathbb{N})$ be a sequence of real-valued $\mathcal{F}$-measurable functions converging to 0, μ-a.e.

(a) Prove that $\lim_{n\to\infty} \int |f_n| d\mu = 0$ if and only if $(f_n : n \in \mathbb{N})$ is UI. (Hint: for one implication use Egoroff's Theorem).

(b) Consider the condition:

$$\text{For every } \epsilon > 0 \text{ there exists } A \in \mathcal{F} \text{ such that } 0 < \mu(A) < \epsilon. \qquad (*)$$

Prove the following:

i. If $(*)$ holds, then there exists a sequence $(g_n : n \in \mathbb{N})$ of nonnegative functions in $L^1(\mu)$ such that $\lim_{n\to\infty} g_n = 0$ μ-a.e., and $\lim_{n\to\infty} \int g_n d\mu = 0$ but there is no $g \in L^1(\mu)$ such that $\sup_{n \in \mathbb{N}} g_n(x) \leq g$, μ-a.e. .

ii. If $(*)$ fails, we have that whenever $(g_n : n \in \mathbb{N})$ is a sequence of functions in $L^1(\mu)$ such that $\lim_{n\to\infty} g_n = 0$ μ-a.e., and $\lim_{n\to\infty} \int g_n d\mu = 0$, then there exists some $N \in \mathbb{N}$ such that $\sup_{n \geq N} g_n(x) \in L^1(\mu)$.

2. Let $f \in L^1(m) \cap L^\infty(m)$. Define a function φ on $\mathbb{R}$ by letting

$$\varphi(x) := \int f(x-t) f(t) dm(t).$$

(a) Prove that the integral defining φ is well-defined (that is, for each $x \in \mathbb{R}$, the function $t \to f(x-t)f(t)$, is in $L^1(m)$).

(b) Prove that φ is continuous.

(c) Prove that $\lim_{|x|\to\infty} \varphi(x)$ exists. Find it.

3. Let $\mathcal{B}_{(0,\infty)}$ denote the Borel σ-algebra on $(0, \infty)$.

(a) Show that there is at most one measure ν on $\mathcal{B}_{(0,\infty)}$ which satisfies the following conditions:

i. $\nu\big((1, e]\big) = 1$.

ii. ν is dilation-invariant. That is, $\nu(cA) = \nu(A)$ for every $c > 0$ and $A \in \mathcal{B}_{(0,\infty)}$ (here, as usual, $cA := \{ca : a \in A\}$).

(Hint: observe that $\nu\big((a, b]\big) = \nu\big((1, \frac{b}{a}]\big)$ and that for any $t > 1$ and $l \in \mathbb{N}$, $(1, t] = \bigcup_{j=0}^{l-1} t^{j/l} \left(1, t^{1/l}\right])$

(b) Assuming that the measure from part (a) exists and is absolutely continuous with respect to the restriction of the Lebesgue measure to $\mathcal{B}_{(0,\infty)}$, find the Radon-Nikodym derivative. Was the assumption correct ?

4. Prove Minkowski's inequality for $L^p(\mu)$, $p \in [1, \infty]$.