Measure and Integration (Math 5111) Aug 2009
Qualifying Exam

ID: ______________
Last: ______________  First: ______________

- Do Problem 5 and any three (complete, not combinations of parts) of Problems 1-4. Mark on this form the problem NOT to be graded.
- Results proved in the textbook/s (Folland, Royden, Rudin or Dudley) should be applied without proof. In this case you have to state accurately the result you are using.
- Below \( m \) denotes the Lebesgue measure on \( \mathbb{R} \).

1. (a) Suppose that \( \{ \mu_n : n \in \mathbb{N} \} \) is a sequence of \( \sigma \)-finite measures on the measurable space \((X, \mathcal{F})\).
   For \( A \in \mathcal{F} \) let \( \nu(A) = \sum_{n=1}^{\infty} \mu_n(A) \). Prove that \( \nu \) is a measure on \((X, \mathcal{F})\).
   
   (b) Let \( \nu \) be the measure from part (a). Prove that one can write \( \nu = \sum_{n=1}^{\infty} \rho_n \), where \( \{ \rho_n : n \in \mathbb{N} \} \) is a sequence of measures on \((X, \mathcal{F})\) satisfying all of the following
   
   (i) For all \( n \in \mathbb{N} \), \( \rho_n \ll \mu_n \);
   (ii) If \( n > 1 \), then \( \rho_k \perp \rho_n \) for all \( k < n \);
   (iii) For all \( n \in \mathbb{N} \), \( \frac{d\rho_n}{d\mu} \in \{0, 1\} \), \( \nu \)-almost everywhere.

2. (a) Let \((X, \mathcal{F}, \mu)\) be a measure space. Suppose that \( \{ f_n : n \in \mathbb{N} \} \) is a sequence of functions in \( L^1(X, \mathcal{F}, \mu) \) which satisfy \( |f_n| \leq h \) for some \( h \in L^1(X, \mathcal{F}, \mu) \) and \( \lim_{n \to \infty} f_n = f \) in measure. Prove that \( \lim_{n \to \infty} \int |f_n - f| \, d\mu = 0 \).
   
   (b) Suppose \( f : [0, \infty) \to [0, \infty) \) is a three times differentiable nonnegative function, which satisfies \( f(0) = f'(0) = 0 \). Assume further that \( f''(0) > 0 \) and \( f^{(3)} \geq 0 \). Prove

   \[
   \lim_{M \to \infty} \sqrt{M} \int_{[0,\infty)} e^{-Mf(x)} \, dm = \frac{c}{\sqrt{f''(0)}}, \quad \text{where} \quad c = \int_{[0,\infty)} e^{-\zeta^2/2} \, dm(\zeta) = \sqrt{\frac{\pi}{2}}.
   \]

3. Let \( f \in L^1(X, \mathcal{F}, \mu) \) and assume further \( \mu(X) < \infty \). Consider the function

   \[
   g(c) = \int |f(x) - c| \, dm.
   \]

   (a) Prove that \( g \) is absolutely continuous on \( \mathbb{R} \) and \( \lim_{|c| \to \infty} g(c) = \infty \).
   (b) Find \( g'(c) \), and prove that \( g(c_0) = \min_{c \in \mathbb{R}} g(c) \) if and only if \( \mu(\{ x : f(x) < c_0 \}) = \mu(\{ x : f(x) > c_0 \}) \) (such a \( c_0 \) is called a median).

4. (a) Assume that \( f \) an absolutely continuous function on \( \mathbb{R} \) satisfying \( f(0) = 0 \) and \( f' \in L^p(m) \) for some \( p > 1 \). Prove that for all \( g \in L^q(m) \), we have

   \[
   \int_0^1 |f| \, dm \leq \left( \frac{1}{p} \right)^{1/p} \left( \int_0^1 |f'|^p \, dm \right)^{1/p} \left( \int_0^1 |g|^q \, dm \right)^{1/q}.
   \]

   Here \( q \) is the conjugate exponent \( \frac{1}{p} + \frac{1}{q} = 1 \).

   (Hint: use the assumptions on \( f \) to express it in terms of \( f' \))
   (b) State and prove the analog of the inequality of part (a) for the case \( p = 1 \).
5. True/False. Determine whether each of the above is true or false (not always true). In the former case, prove. In the latter case, provide a counterexample.

Note: the parts are not related.

(a) The function
\[ f(x) = \begin{cases} \frac{1}{q} & x = p/q, \ p \in \mathbb{Z}^+, q \in \mathbb{N} \text{ are relatively prime} \\ 0 & \text{otherwise.} \end{cases} \]
is Riemann integrable on \([0, 1]\).

(b) Suppose \( f \) is a measurable function on the product measure space \((X \times Y, \mathcal{F} \times \mathcal{G}, \mu \times \nu)\), and that both iterated integrals \( \int_Y \int_X f(x,y) d\mu(x) d\nu(y) \), \( \int_Y \int_X f(x,y) d\nu(y) d\mu(x) \) are well defined, finite and equal to 0. Then \( f \in L^1(\mu \times \nu) \).

(c) Suppose that the sequence of measurable functions \( \{f_n : n \in \mathbb{N}\} \) satisfies \( \lim_{n \to \infty} \int f_n g dm = 0 \) for all \( g \in L^1(m) \). Then \( \lim_{n \to \infty} f_n = 0 \) in measure.

(d) Suppose that \((X, \mathcal{F}, \mu)\) is a finite measure space. Then \( 1 \leq p_1 < p_2 \) implies \( L^{p_1}(\mu) \supset L^{p_2}(\mu) \).

(e) If \( \{f_n : n \in \mathbb{N}\} \) is a sequence of nonnegative Lebesgue-measurable functions then \( \lim \sup_{n \to \infty} \int f_n dm \leq \int \lim \sup_{n \to \infty} f_n dm \).