1. Prove
\[
\sum_{k=1}^{\infty} \frac{1}{(p+k)^2} = \int_0^1 \frac{x^p \log x^{-1}}{1-x} \, dx
\]
for \( p > 1 \). Justify each step, in particular indicate why certain improper Riemann integrals are Lebesgue integrals.

2. Give an example of a sequence of functions that converges in \( L_1 \) but not almost everywhere (a.e.). Show that, on the other hand, if \( f_n, n \in \mathbb{N} \), and \( f \) are in \( L_1 \) and \( f_n \to f \) in \( L_1 \) fast enough so that \( \sum_n \int |f_n - f| < \infty \), then \( f_n \to f \) a.e.

3. Let \((X, \mathcal{M}, \mu)\) be a \( \sigma \)-finite measure space and let \( f \in L_1(X, \mu) \), \( f \geq 0 \). Show that the subgraph of \( f \),
\[ G_f := \{(x, y) \in X \times [0, \infty] : y \leq f(x)\} \]
is \( \mathcal{M} \times B_\mathbb{R} \)-measurable and
\[ (\mu \times m)(G_f) = \int f \, d\mu, \]
where \( m \) is Lebesgue measure (the integral of a non-negative function is the area under its graph, above the 'x-axis').

4. Let
\[ f(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\frac{x^2 \sin x}{x} & \text{if } 0 < x \leq 1
\end{cases} \]
Determine whether this function is of bounded variation on \([0, 1]\) and whether it is absolutely continuous on \([0, 1]\). Determine the same on \([\delta, 0]\) for any \( 0 < \delta < 1 \). Justify your answers.

5. Show that if \( f \in L_p(\mathbb{R}) \cap L_{\infty}(\mathbb{R}) \) for some \( p \geq 1 \) then \( f \in L_q(\mathbb{R}) \) for all \( q > p \) and
\[ \|f\|_\infty = \lim_{q \to \infty} \|f\|_q. \]

Hint: It suffices to consider \( \|f\|_\infty = 1 \) and \( |f(x)| \leq 1 \) for all \( x \), and, in this case, it will help to look at the functions \( f_\delta := (|f| \wedge (1 - \delta)) / (1 - \delta) \) for suitable \( 0 < \delta < 1 \) for the inequality in one direction.