MATH313 – Preliminary Examination.

August 20, 2007

Instructions: Answer two out of the four questions. You do not have to prove results which you rely upon, just state them clearly!

Q1) (a) Suppose that $A^{(1)} = A$ is an invertible $n \times n$ matrix and that the Gaussian elimination algorithm with partial pivoting applied to $A^{(1)}$ produces the upper triangular matrix $A^{(n)}$. As usual, let $A^{(k)}$ be the renamed $A^{(k)}$ following any necessary row interchanges before the $k$–th major step of the elimination so that

$$a_{i,j}^{(k+1)} = \begin{cases} a_{i,j}^{(k)}, & \text{when } i \leq k, \ 1 \leq j \leq n, \\ 0, & \text{when } i \geq k+1, \ 1 \leq j \leq k, \\ a_{i,j}^{(k)} - a_{i,k}^{(k)} a_{k,j}^{(k)}/a_{k,k}^{(k)}, & \text{when } i, j \geq k+1. \end{cases}$$

Show that the total number of multiplication and division operations needed to reduce $A^{(1)}$ to $A^{(n)}$ is $(n^3 - n)/3$. [Hint: Recall that $\sum_{i=1}^{n} i^2 = n(n + 1)(2n + 1)/6$.]

b) Suppose that all the leading principal minors of $A$ are positive. Show that $A$ has an LU–factorization with unit diagonal entries in $L$ and positive diagonal entries in $U$.

c) Suppose now that no partial pivoting is necessary and that $A^{(1)} = (a_{i,j}^{(1)})$ is tridiagonal, that is, $a_{i,j}^{(1)} = 0$ when $|i - j| > 1$, $1 \leq i, j \leq n$. Show that each of $A^{(1)}, \ldots, A^{(n)}$ is tridiagonal.
d) Suppose that $A$ is an $n \times n$ invertible matrix which admits an LU–factorization without pivoting. Partition $A$ into:

$$
A = \begin{pmatrix}
A_{1,1} & A_{1,2} \\
A_{2,1} & A_{2,2}
\end{pmatrix},
$$

with $A_{1,1}$ being a $(k - 1) \times (k - 1)$ matrix. Knowing that $A_{1,1}$ is invertible (why?), show that the current active array which is the $(n-k+1) \times (n-k-1)$ matrix $A_k = (a_{i,j}^{(k)})$, $i,j = k, \ldots, n$ is given by:

$$
A_k = A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}.
$$

Assume now that in addition to $A$ being invertible, $A$ is Hermitian. Use this formula to deduce that $A_k$ is also Hermitian, $k = 1, \ldots, n$.

Q3) (a) Prove: A quadrature formula $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ that uses the $n+1$ distinct nodes $x_0, \ldots, x_n$ and is exact of order at least $n$ is interpolatory, that is,

$$
\alpha_k = \int_a^b L_k(x)dx, \ k = 0, \ldots, n,
$$

where

$$
L_k(x) = \frac{\prod_{j=0}^{n} (x - x_j)}{\prod_{j \neq k}^{n} (x_k - x_j)}, \ k = 0, \ldots, n.
$$

(b) The Legendre polynomial of degree $n$ is defined by

$$
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,
$$

with $P_0(x) \equiv 1$. Calculate explicitly $P_1, \ldots, P_4$. Prove (verify) that for $k = 0, 1, \ldots, n - 1$,

$$
\int_{-1}^{1} x^k P_n(x)dx = 0.
$$

(c) Use part (b) to conclude that $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$, when $m \neq n$, and that $\int_{-1}^{1} P_n^2(x)dx = 2/(2n + 1)$. 

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Q4) (a) Derive the recurrence relation \( T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \) for the Tchebyshev polynomials:

\[
T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, \ldots
\]

and prove that \( \hat{T}_n(x) = (1/2^{n-1})T_n(x) \) is a monic polynomial (that is, the leading coefficient is 1).

(b) Prove that \( \hat{T}_n(x) \) has minimal infinity norm among all monic polynomials of degree \( n \) on the interval \([-1, 1]\). Moreover, show that \( \| \hat{T}_n(x) \|_{\infty} = 1/2^{n-1} \), where \( \| \cdot \|_{\infty} \) denotes the maximum norm of a function on the interval \([-1, 1]\).

(c) Obtain that \( p(x) \approx 0.98516 + 0.11961x \) is the best approximation polynomial of order at most 1 to the function \( f(x) = \sqrt{1 + (1/4)x^2} \) over the interval \([0, 1] \)