MATH313 – Preliminary Examination.

August 20, 2007

Instructions: Answer two out of the four questions. *You do not have to prove results which you rely upon, just state them clearly !*

Q1) (a) Suppose that $A^{(1)} = A$ is an invertible $n \times n$ matrix and that the Gaussian elimination algorithm with partial pivoting applied to $A^{(1)}$ produces the upper triangular matrix $A^{(n)}$. As usual, let $A^{(k)}$ be the renamed $A^{(k)}$ following any necessary row intrechanges before the k–th major step of the elimination so that

$$a_{i,j}^{(k+1)} = \begin{cases} a_{i,j}^{(k)}, & \text{when } i \leq k,\ 1 \leq j \leq n, \\ 0 & \text{when } i \geq k+1,\ 1 \leq j \leq k, \\ a_{i,j}^{(k)} - a_{i,k}^{(k)} a_{k,j}^{(k)} / a_{k,k}^{(k)}, & \text{when } i, j \geq k+1. \end{cases}$$

Show that the total number of multiplication and division operations needed to reduce $A^{(1)}$ to $A^{(n)}$ is $(n^3 - n)/3$. [Hint: Recall that $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.]

b) Suppose that all the leading principal minors of A are positive. Show that A has an LU–factorization with unit diagonal entries in L and positive diagonal entries in U.

c) Suppose now that no partial pivoting is necessary and that $A^{(1)} = \left(a_{i,j}^{(1)}\right)$ is tridiagonal, that is, $a_{i,j}^{(1)} = 0$ when $|i-j| > 1,\ 1 \leq i, j \leq n$. Show that each of $A^{(1)}, \ldots, A^{(n)}$ is tridiagonal.

d) Suppose that A is an $n \times n$ invertible matrix which admits an LU–factorization without pivoting. Partition A into:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ \\ A_{2,1} & A_{2,2} \end{pmatrix},$$

with $A_{1,1}$ being a $(k-1) \times (k-1)$ matrix. Knowing that $A_{1,1}$ is invertible (why?), show that the current active array which is the $(n-k+1) \times (n-k-1)$ matrix $A_k = \left(a_{i,j}^{(k)}\right)$, $i, j = k, \ldots, n$ is given by:

$$A_k = A_{2,2} - A_{2,1} A_{2,2}^{-1} A_{1,2}.$$

Assume now that in addition to A being invertible, A is Hermitian. Use this formula to deduce that A_k is also Hermitian, $k = 1, \ldots, n$.

Q3) (a) Prove: A qudrature formula $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ that uses the $n+1$ distinct nodes $x_0, \ldots, x_n$ and is exact of order at least n is interpolatory, that is,

$$\alpha_k = \int_a^b L_k(x) dx, \quad k = 0, \ldots, n,$$

where

$$L_k(x) = \frac{\prod_{\substack{j=0 \\ j \neq k}}^{n} (x - x_j)}{\prod_{\substack{j=0 \\ j \neq k}}^{n} (x_k - x_j)}, \quad k = 0, \ldots, n.$$

(b) The Legendre polynomial of degree n is defined by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n,$$

with $P_0(x) \equiv 1$. Calculate explicitly $P_1, \ldots, P_4$. Prove (verify) that for $k = 0, 1, \ldots, n-1$,

$$\int_{-1}^{1} x^k P_n(x) dx = 0.$$

(c) Use part (b) to conclude that $\int_{-1}^{1} P_n(x) P_m(x) dx = 0$, when $m \neq n$, and that $\int_{-1}^{1} P_n^2(x) dx = 2/(2n+1)$.

Q4) (a) Derive the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the Tchebyshev polynomials:

$$T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, ...$$

and prove that $\hat{T}_n(x) = (1/2^{n-1})T_n(x)$ is a monic polynomial (that is, the leading coefficient is 1).

(b) Prove that $\hat{T}_n(x)$ has minimal infinity norm among all monic polynomials of degree n on the interval $[-1, 1]$. Moreover, show that $\|\hat{T}_n(x)\|_\infty = 1/2^{n-1}$, where $\|\cdot\|_\infty$ denotes the maximum norm of a function on the interval $[-1, 1]$.

(c) Obtain that $p(x) \approx 0.98516 + .11961x$ is the best approximation polynomial of order at most 1 to the function $f(x) = \sqrt{1 + (1/4)x^2}$ over the interval $[0, 1]$