MATH 313 August 2004 PRELIMINARY EXAMINATION

INSTRUCTIONS: Answer three out of four questions. You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) **(a)** Prove: An $n \times n$ matrix $A = (a_{i,j})$ admits an LU-factorization $A = LU$ without pivoting and with invertible factors L and U if and only if for $k = 1, \ldots, n$, the leading principal submatrices of A of order k are all invertible.

(b) Let A be an $n \times n$ invertible matrix that admits an LU-factorization without pivoting. Show that such a factorization is unique; namely, if $A = L_1U_1 = L_2U_2$, where L_1 and L_2 are lower triangular matrices with diag($L1$) =diag(L_2) $= I$ and where U_1 and U_2 are upper triangular, then $L_1 = L_2$ and $U_1 = U_2$.

(c) Suppose that A is a real $n \times n$ symmetric invertible matrix which admits an LU-factorization $A = LU$, with a lower triangular matrix L such that diag(L) $= I$, and with an upper triangular matrix U having positive diagonal entries. Show that A admits a factorization $A = \tilde{L}\tilde{L}^T$.

Q2) Let $w(x)$ be a positive continuous function on $[a, b]$. For $j = 1, 2, ...,$ let $p_j(x)$ be the corresponding monic orthogonal polynomial of degree j, i.e.,

$$p_j(x) = x^j + a_1x^{j-1} + \cdots + a_j,$$

such that $(p_j, p_k) = \int_a^b w(x)p_j(x)p_k(x)dx = 0$ if $j \neq k$. In particular $p_0(x) = 1$.

(a) Prove that the roots $x_1, .., x_n$ of $p_n(x)$ are real, simple and lie in (a, b).

(b) Prove that $p_n(x)$ satisfy a three term recurrence relation, i.e.,

$$p_{i+1}(x) = (x - \delta_{i+1})p_i(x) - \gamma_{i+1}^2 p_{i-1}(x), \quad \imath \geq 0,$$

where $p_{i-1} = 0, \quad \gamma_1 = 0$, and

$$\delta_{i+1} = \frac{(xp_i, p_i)}{(p_i, p_i)}, \quad i \geq 0, \quad \gamma_{i+1}^2 = \frac{(p_i, p_i)}{(p_{i-1}, p_{i-1})}, \quad i \geq 1.$$

(c) For $a = -1; b = 1; w(x) = 1$; find $p_1(x)$ and $p_2(x)$.

Q3) **(a)** Derive the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the Tchebyshev polynomials:

$$T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, ...$$

and prove that $\hat{T}_n(x) = (1/2^{n-1})T_n(x)$ is monic (that is, the leading coefficient is 1).

(b) Prove that $\hat{T}_n(x)$ has minimal infinity norm on the interval $[-1,1]$,

$$\|\hat{T}_n(x)\|_\infty = max_{-1\le x\le 1}|\hat{T}_n(x)|$$

among all monic polynomials of degree n on the interval $[-1,1]$. Moreover, show that $\|\hat{T}_n(x)\|_\infty = 1/2^{n-1}$.

(c) Find $n+1$ points $\{x_i\}$ in the interval $[0,1]$ that minimize

$$\Delta^n_{[0,1]} = \max_{0\le\xi\le 1}\prod_{i=0}^{n}(\xi - x_i)$$

(d) Find the (minimal) value $\Delta^n_{[0,1]}$ itself (Remark: for the interval $[0,1]$ that we consider here such a minimal value should differ from $\Delta^n_{[-1,1]}$ corresponding to the interval $[-1,1]$ which should be known to you from the (b)).

Q4) (a) Let $N = 2M+1$ and consider

$$\Psi(x) = \frac{A_0}{2} + \sum_{h=1}^{M}(A_h \cos hx + B_h \sin hx) \tag{1}$$

and

$$p(x) = \beta_0 + \beta_1 e^{ix} + \beta_2 e^{2ix} + \ldots + \beta_{N-1}e^{(N-1)ix}$$

Assume that $\Psi(x)$ and $p(x)$ agree at the N points

$$x_k = 2\pi k/N, \qquad k = 0,1,\ldots,N-1$$

i.e.,

$$\Psi(x_k) = p(x_k), \qquad k = 0,1,\ldots,N-1.$$

Use the relation between e^{x_k} and $e^{x_{N-k}}$ to find the matrix R such that

$$\begin{bmatrix} A_0 & A_1 & A_2 & \cdots & A_M & B_M & \cdots & B_2 & B_1 \end{bmatrix} \cdot R = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{N-1} \end{bmatrix} \tag{2}$$

(b) Explain why the matrix R in (2) is invertible, and use the uniqueness of the interpolation polynomial to show that the trigonometric polynomial (1) satisfying

$$\Psi(x_k) = y_k, \quad y_k \in \mathbb{C}, \quad k = 0,...,N-1. \tag{3}$$

is unique.

(c) Explain how to solve the trigonometric interpolation problem in (3) with the help of (2) via the inverse FFT (provide the definition for the DFT).