INSTRUCTIONS: Answer three out of four questions. You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) (a) Prove: An $n \times n$ matrix $A = (a_{i,j})$ admits an LU-factorization $A = LU$ without pivoting and with invertible factors $L$ and $U$ if and only if for $k = 1, \ldots, n$, the leading principal submatrices of $A$ of order $k$ are all invertible.

(b) Let $A$ be an $n \times n$ invertible matrix that admits an LU-factorization without pivoting. Show that such a factorization is unique; namely, if $A = L_1U_1 = L_2U_2$, where $L_1$ and $L_2$ are lower triangular matrices with diag($L_1$) = diag($L_2$) = $I$ and where $U_1$ and $U_2$ are upper triangular, then $L_1 = L_2$ and $U_1 = U_2$.

(c) Suppose that $A$ is a real $n \times n$ symmetric invertible matrix which admits an LU-factorization $A = LU$, with a lower triangular matrix $L$ such that diag($L$) = $I$, and with an upper triangular matrix $U$ having positive diagonal entries. Show that $A$ admits a factorization $A = \tilde{L}\tilde{L}^T$.

Q2) Let $w(x)$ be a positive continuous function on $[a, b]$. For $j = 1, 2, \ldots$, let $p_j(x)$ be the corresponding monic orthogonal polynomial of degree $j$, i.e.,

$$p_j(x) = x^j + a_1x^{j-1} + \cdots + a_j,$$

such that $(p_j, p_k) = \int_a^b w(x)p_j(x)p_k(x)dx = 0$ if $j \neq k$. In particular $p_0(x) = 1$.

(a) Prove that the roots $x_1, \ldots, x_n$ of $p_n(x)$ are real, simple and lie in $(a, b)$.

(b) Prove that $p_n(x)$ satisfy a three term recurrence relation, i.e.,

$$p_{i+1}(x) = (x - \delta_{i+1})p_i(x) - \gamma_i^2 p_{i-1}(x), \quad i \geq 0,$$

where $p_{i-1} = 0, \gamma_1 = 0$, and

$$\delta_{i+1} = \frac{(xp_i, p_i)}{(p_i, p_i)}, \quad i \geq 0, \quad \gamma_i^2 = \frac{(p_i, p_i)}{(p_{i-1}, p_{i-1})}, \quad i \geq 1.$$

(c) For $a = -1; b = 1; w(x) = 1$; find $p_1(x)$ and $p_2(x)$.

Q3) (a) Derive the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the Tchebyshev polynomials:

$$T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, \ldots,$$

and prove that $\hat{T}_n(x) = (1/2^{n-1})T_n(x)$ is monic (that is, the leading coefficient is 1).
(b) Prove that $\hat{T}_n(x)$ has minimal infinity norm on the interval $[-1, 1]$,

$$
\| \hat{T}_n(x) \|_\infty = \max_{-1 \leq x \leq 1} |\hat{T}_n(x)|
$$

among all monic polynomials of degree $n$ on the interval $[-1, 1]$. Moreover, show that $\| \hat{T}_n(x) \|_\infty = 1/2^{n-1}$.

(c) Find $n + 1$ points $\{x_i\}$ in the interval $[0, 1]$ that minimize

$$
\Delta_n^{[0,1]} = \max_{0 \leq \xi \leq 1} \prod_{i=0}^{n} (\xi - x_i)
$$

(d) Find the (minimal) value $\Delta_n^{[0,1]}$ itself (Remark: for the interval $[0, 1]$ that we consider here such a minimal value should differ from $\Delta_n^{[-1,1]}$ corresponding to the interval $[-1, 1]$ which should be known to you from the (b)).

Q4) (a) Let $N = 2M + 1$ and consider

$$
\Psi(x) = \frac{A_0}{2} + \sum_{h=1}^{M} (A_h \cos hx + B_h \sin hx)
$$

and

$$
p(x) = \beta_0 + \beta_1 e^{ix} + \beta_2 e^{2ix} + \ldots + \beta_{N-1} e^{(N-1)ix}
$$

Assume that $\Psi(x)$ and $p(x)$ agree at the $N$ points

$$
x_k = 2\pi k / N, \quad k = 0, 1, \ldots, N - 1
$$

i.e.,

$$
\Psi(x_k) = p(x_k), \quad k = 0, 1, \ldots, N - 1.
$$

Use the relation between $e^{ix}$ and $e^{xN-k}$ to find the matrix $R$ such that

$$
\begin{bmatrix}
A_0 & A_1 & A_2 & \cdots & A_M & B_M & \cdots & B_2 & B_1
\end{bmatrix} \cdot R = \begin{bmatrix}
\beta_0 & \beta_1 & \cdots & \beta_{N-1}
\end{bmatrix} \quad (2)
$$

(b) Explain why the matrix $R$ in (2) is invertible, and use the uniqueness of the interpolation polynomial to show that the trigonometric polynomial (1) satisfying

$$
\Psi(x_k) = y_k, \quad y_k \in \mathbb{C}, \quad k = 0, \ldots, N - 1. \quad (3)
$$

is unique.

(c) Explain how to solve the trigonometric interpolation problem in (3) with the help of (2) via the inverse FFT (provide the definition for the DFT).