INSTRUCTIONS: Answer three out of four questions. You do not have to prove results which you rely upon, just state them clearly.

Q1) (a) Suppose that \( p(x) \) is a polynomial of degree at most \( n \) which has \( n + 1 \) distinct roots. Show that \( p(x) \equiv 0 \). Use this result to show that the polynomial \( p_n \), of order at most \( n \), which interpolates a function \( f \) at \( n + 1 \) distinct points \( x_0, \ldots, x_n \) is unique. [Assume that the values which \( f \) takes at these points are \( f_0, \ldots, f_n \), respectively.]

(b) Suppose that \( f \in C^{n+1}[a,b] \) and that \( x_0, \ldots, x_n \) are \( n + 1 \) distinct points in the interval. Let \( p_n \) be the interpolation polynomial for \( f \) on \( x_0, \ldots, x_n \). Let \( e_n(x) = f(x) - p_n(x) \) denote the error function on \([a,b]\). Show that for each point \( x \in [a,b] \), there is a point \( \xi_x \in (a,b) \) such that
\[
e_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n).
\]

(c) A function \( f \) is defined on the interval \([0,1]\) and its derivatives satisfy that \( |f^{(m)}(x)| \leq m! \), for all \( x \in [0,1] \) and for all \( m = 0, 1, 2, \ldots \). For any \( 0 < q < 1 \), let \( p_n(x), n \geq 0, \) be the interpolation polynomial of degree at most \( n \) which interpolates \( f \) at \( x_0 = 1, x_1 = q, x_2 = q^2, \ldots, x_n = q^n \). Show that
\[
limit_{n \to \infty} p_n(0) = f(0).
\]
Taking \( q = 1/2 \) and \( n = 10 \), find an upper estimate on \( |p_{10}(0) - f(0)| \).

Q2) The following compactness theorem is known: Let \( V \) be a finite dimensional normed vector space and \( W \) be a closed subset of \( V \). If there exists a constant \( M > 0 \) such that \( \|w\| \leq M \) for all \( w \in W \), then any sequence in \( W \) has a convergent subsequence.

Define \( P_n \) to be the vector space of polynomials of degree at most \( n \) and \( \|f\| = \max_{0 \leq x \leq 1} |f(x)| \) for any continous function \( f \in C[0,1] \).

(a) Show that for any \( f \in C[0,1] \), there exists a polynomial \( p^* \in P_n \) which minimizes the uniform norm of \( \|f - q\| \) for any \( q \in P_n \).

(Hint: let \( \inf_{w \in W} \|w - f\| = \alpha \). Then there exists a sequence \( \{w_i\} \subset W \) such that \( \|w_i - f\| \to \alpha \) as \( i \to \infty \). The sequence \( \{w_i\} \) is called a minimizing sequence.)

(b) Define a set on rational functions
\[
R_{n,m} = \{ \frac{p(x)}{q(x)} : p \in P_n \text{ and } q \in P_m \text{, } \|q\| = 1 \text{, } q > 0 \text{ on } [0,1], \}
\]
\( p \) and \( q \) have no common factors.}.
Our Goal: Given $f \in C[0, 1]$, prove the existence of $r^* \in R_{n,m}$ such that it minimizes the uniform norm of $\|f - r\|$ for any $r \in R_{n,m}$.

Let $p_i/q_i$ be a minimizing sequence. Show that there exists a constant $M$ such that $\|q_i\|, \|p_i/q_i\|$ and $\|p_i\|$ are all bounded by $M$ for all $i$.

(c) By Q2a, we can assume that (a subsequence of) $p_i$ and $q_i$ converge to $p \in P_n$ and $q \in P_m$, respectively. Explain why $q \geq 0$ and can have at most finite number of roots of even multiplicity in $[0, 1]$.

(d) Let $z$ be a root of $q$, explain why $z$ has to be a root of $p$ of at least the same multiplicity. (Hint: $\|p_i/q_i\| \leq M$ from part Q2b). Hence try to finish the proof for our goal stated in Q2b.

Q3) (a) Recall that the 1–norm of a vector $x = (x_1, \ldots, x_n) \in C^n$ is given by $\|x\|_1 = \sum_{i=1}^n |x_i|$. Show that for $n \times n$ matrix $A = (a_{i,j}) \in C^{n,n}$, the 1–matrix norm induced by the 1–vector norm, that is, by

$$\|A\|_1 = \max_{\|x\|_1 = 1, x \in C^n} \|Ax\|_1,$$

is given by

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{i,j}|.$$

(b) Recall that for a matrix $B = (b_{i,j}) \in C^{n,n}$, $\|B\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |b_{i,j}|$ and that if $B$ is invertible, then $\text{cond}_\infty(B) := \|B\|_\infty \|B^{-1}\|_\infty$.

Suppose now that $A = (a_{i,j}) \in C^{n,n}$ is an invertible matrix with $\sum_{j=1}^n |a_{i,j}| = 1$, $1 \leq i \leq n$. Show, first, that if $D$ is any invertible diagonal matrix, then $\|DA\|_\infty = \|D\|_\infty$ and use this to show that

$$\text{cond}_\infty(A) \leq \text{cond}_\infty(DA),$$

Discuss the following problem: Can the numerical stability of solving the system $Ax = b$, where $A$ is as above, be improved by scaling the rows of the matrix $A$ and the vector $b$ by a diagonal matrix $D$, namely, by solving instead the system $A'x = b'$, where $A' = DA$ and $b' = Db$, for some invertible diagonal matrix $D$.

Q4) (a) Consider the uniform partition of the interval $[0, 2\pi]$,

$$x_k = \frac{2\pi k}{N}, \quad k = 0, \ldots, N-1, \quad N = 2M + 1.$$ 

Show that there exists a unique trigonometric polynomial

$$\Psi(x) = \frac{A_0}{2} + \sum_{h=1}^M (A_h \cos(hx) + B_h \sin(hx))$$

such that

$$\Psi(x_k) = y_k, \quad y_k \in C, \quad k = 0, \ldots, N-1.$$ 

(b) Show that if $y_k, k = 0, \ldots, N-1$ are real numbers, then $A_h$ and $B_h$ are also real numbers.