Topology Prelim, January 2014

1. Let X, Y be topological spaces, Y Hausdorff, and let $A \subset X$ be a non-empty set.

   (a) Suppose that $f : A \to Y$ is continuous, where A is equipped with the subspace topology. Prove that if there exists a continuous extension of f to $\overline{A}$, it is unique.

   (b) Assume that A is connected in the subspace topology. Prove that $\overline{A}$ is connected in the subspace topology.

2. Let $\mathcal{S}$ denote the standard topology on $\mathbb{R}$ and let $\mathcal{T}$ be the topology on $\mathbb{R}$ generated by the intervals $[a, b)$, where $a \in \mathbb{Q}$ and $b \in \mathbb{R}$.

   (a) Suppose that $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{S})$ is a function. Show that f is continuous if and only if the function $g : (\mathbb{R}, \mathcal{S}) \to (\mathbb{R}, \mathcal{S})$ given by $g(x) = f(x)$ is right-continuous at all rational points and continuous at all irrational points.

   (b) Is $(\mathbb{R}, \mathcal{T})$ metrizable ?

3. Suppose that X is a topological space homeomorphic to an open subset of a compact Hausdorff space. Prove that X is locally compact (=every point has a neighborhood contained in a compact set).

4. The real projective plane P^2 is the topological space of lines in $\mathbb{R}^3$ passing through the origin. One construction of P^2 is as a quotient space of the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ with the subspace topology, obtained by identifying antipodal points.

   (a) Prove that P^2 is compact Hausdorff and that every point in P^2 has a neighborhood homeomorphic to the open unit ball in $\mathbb{R}^2$.

   (b) Prove that the quotient map is a covering map.

   (c) Find a path in S^2 whose image under the quotient map generates the fundamental group for P^2 based at the image of $(1, 0, 0)$ under the quotient map.

5. Show that $\mathbb{R}^3$ is not homeomorphic to $\mathbb{R}^2$.

6. Let X be the subspace of $\mathbb{R}^3$ equal to the union of the unit sphere with the three line segments $\{(0, 0, z) : |z| \leq 1\} \cup \{(0, y, 0) : |y| \leq 1\} \cup \{(x, 0, 0) : |x| \leq 1\}$. Compute the fundamental group of X based at $(1, 0, 0)$.