PRELIM 5310 PRELIM (Topology)
January 2012

Justify all your steps rigorously. You may use any results that you know, unless the question asks you to prove essentially the same result.

1. Decide whether each of the following statements is correct. If yes, give a proof, otherwise, give a counterexample (without proof). Here X and Y are topological spaces.

   (a) Let $C \subset X$ be a closed subset. Then C is equal to the closure of its interior: $C = \overline{\text{int}C}$.

   (b) If $f\colon (X,x) \to (Y,y)$ is surjective and continuous, then the induced map $f_*\colon \pi_1(X,x) \to \pi_1(Y,y)$ is surjective.

2. Assume that X, Y are connected. Prove that $X \times Y$ is connected.

3. (a) Let X be a Hausdorff space, and let $C_1, C_2 \subset X$ be disjoint compact subsets. Prove that there exist disjoint open subsets $U_1, U_2 \subset X$ with $C_1 \subset U_1$ and $C_2 \subset U_2$.

   (b) Assume that X is compact and Hausdorff, and assume that $f\colon X \to Y$ is a continuous and closed surjective map. Prove that Y is Hausdorff.

4. Let $X \subset \mathbb{R}^3$ denote the union of the unit sphere in $\mathbb{R}^3$ with the unit disc in the xy-plane, i.e.:

   $$X = \{(x,y,z) \,|\, x^2+y^2+z^2 = 1\} \cup \{(x,y,z) \,|\, x^2+y^2 \le 1, z=0\}.$$

   Compute $\pi_1(X)$.

5. Let X be a compact and connected manifold, and let $p\colon \widetilde{X} \to X$ be a connected covering.

   (a) Prove that for $x, x' \in X$, the sets $p^{-1}(x)$ and $p^{-1}(x')$ have the same cardinality. (*Note:* Two sets A, B have the same cardinality iff there is a bijection $A \to B$.)

   (b) Prove that $\widetilde{X}$ is compact if and only if the set $p^{-1}(x)$ is finite.

6. Assume that (X, x_0) is connected, locally path-connected and semilocally simply-connected. Let $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a connected covering. Show that the stabilizer of $\tilde{x}_0$ of the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is equal to the subgroup in $\pi_1(X, x_0)$ corresponding to p under the Galois correspondence. (*Note*: Given an action of a group G on a set S, the stabilizer of an element $s \in S$ is the subgroup $\{g \in G \,|\, g.s = s\}$.)