5310 PRELIM
Introduction to Geometry and Topology
January 2011

You may use any result that we proved in class (unless the question directly asks you to prove this result!).

1. Let $f_1, f_2 \colon X \to Y$ be continuous maps from a topological space X to a Hausdorff space Y. Show that the set S of points $\{x \in X \mid f_1(x) = f_2(x)\}$ where f_1 and f_2 are equal is a closed set.

2. (a) Prove the following statement.

   Tube Lemma: Assume that Y is compact, and that U_α for $\alpha \in A$ is a collection of open subsets of $X \times Y$ that covers $\{x\} \times Y$ for some $x \in X$, i.e. $\{x\} \times Y \subset \bigcup_{\alpha \in A} U_\alpha$.

   Then there is a finite subcollection $U_1, \ldots, U_n$ with $U_i = U_{\alpha_i}$ for some $\alpha_i \in A$, and an open set $V \subset X$ such that $U_1, \ldots, U_n$ covers $V \times Y$, i.e. $V \times Y \subset U_1 \cup \cdots \cup U_n$.

   (b) Prove directly from the definition that if X, Y are compact, then $X \times Y$ is compact. *Hint: Use the Tube Lemma.*

3. Assume that X is a path-connected space with basepoint $x_0 \in X$. Let $A \subset X$ be a path-connected subset with $x_0 \in A$. Let $p \colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a base-point preserving covering, and let $\widetilde{A} = p^{-1}(A)$ be the preimage of A.

   Show that if $\widetilde{X}$ is path-connected and if the natural map $i_* \colon \pi_1(A) \to \pi_1(X)$ is surjective, then $\widetilde{A}$ is path-connected.

4. Determine, with proof, the number of connected 2:1-coverings of the wedge sum $S^1 \vee S^1 \vee S^1$.

5. Let $X = [0,1]^2/\sim$ be the quotient of the the unit square $[0,1]^2 \subset \mathbb{R}^2$ modulo the equivalence relation generated by

   $$(t, 0) \sim (1, t) \sim (1 - t, 1) \sim (0, t)$$

   for all $0 \leq t \leq 1$. (One could describe this via the polygon representation $\langle a | aaaa^{-1} \rangle$.)

   Prove that $\pi_1(X) = \mathbb{Z}/2\mathbb{Z}$. Justify your steps carefully.

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   Then there is a finite subcollection $U_1, \ldots, U_n$ with $U_i = U_{\alpha_i}$ for some $\alpha_i \in A$, and an open set $V \subset X$ such that $U_1, \ldots, U_n$ covers $V \times Y$, i.e. $V \times Y \subset U_1 \cup \cdots \cup U_n$.

   (b) Prove directly from the definition that if X, Y are compact, then $X \times Y$ is compact. *Hint: Use the Tube Lemma.*

3. Prove that a compact subset S of a Hausdorff space X is closed. Give an example where this statement fails in case X is not Hausdorff.

4. Show that the product of paths is well-defined on homotopy classes.

5. Let $p\colon X \to Y$ be a covering map, let $f\colon B \to Y$ be a continuous map where B is connected. Prove that if two lifts $f_1, f_2\colon B \to X$ of f agree at a single point $b \in B$, then $f_1 = f_2$.