Topology Prelim August 2010

1. Let X and Y be connected spaces. If A is a proper subset of X and B is a proper subset of Y then $X \times Y - A \times B$ is connected. (A is a proper subset of X if $A \subset X$ and $A \neq X$.)

2. Let X be a first countable space.

   (a) For any set $A \subset X$ and any point $p \in X$, show that $p \in \overline{A}$ if and only if there is a sequence $\{p_n\}_{n=1}^{\infty}$ in A such that $\{p_n\}$ converges to p.

   (b) Show that for any space Y, a map $f : X \longrightarrow Y$ is continuous if and only if f takes convergent sequences in X to convergent sequences in Y.

3. (a) If X is a locally connected space then prove that the components of X are open subsets of X.

   (b) Let $p : X \longrightarrow Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected. (Hint: If C is a component of an open set U of Y, show that $p^{-1}(C)$ is a union of components of $p^{-1}(U)$.)

4. Let X be a Hausdorff space. Suppose that $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ is a collection of compact, connected subsets of X simply ordered by inclusion (that is, for each $\alpha, \beta \in \mathcal{A}$ we have either $A_\alpha \subset A_\beta$ or $A_\beta \subset A_\alpha$). Prove that $\cap_{\alpha \in \mathcal{A}} A_\alpha$ is nonempty and connected.

5. A continuous map $f : X \longrightarrow X$ is called a <u>retraction</u> of X onto $A = f(X)$ if $f \circ f = f$. The image A of f is called a <u>retract</u> of X.

   (a) Prove that any retract of a Hausdorff space is a closed set.

   (b) Let $a \in A$. Show that $f_* : \pi_1(X, a) \longrightarrow \pi_1(A, a)$ is surjective.

6. Let $p : X \longrightarrow Y$ be a covering map, where X and Y are path connected and locally path connected, and let $x_0 \in p^{-1}(y_0)$. Prove the Unique Path Lifting Theorem: Suppose $f : [0, 1] \longrightarrow Y$ is any path with initial point y_0. Then there exists a unique lift $\tilde{f} : [0, 1] \longrightarrow X$ of f such that $\tilde{f}(0) = x_0$.