1. (a) Suppose \( A \) and \( B \) are connected subsets of the space \( X \) and \( A \cap B \neq \emptyset \). Prove that \( A \cup B \) is connected.

(b) Let \( Y \) denote the set of all points in the plane \( \mathbb{R}^2 \) with at least one irrational coordinate. Prove or disprove: \( Y \) is connected.

(c) Let \( \{A_i\} \) be a sequence of connected subspaces of \( \mathbb{R}^2 \) such that \( A_{i+1} \subseteq A_i \) for \( i = 1, 2, 3, \ldots \). Prove or disprove: \( \cap \{A_i | i \geq 1\} \) is connected.

2. Let \( X \) be a compact Hausdorff space.

(a) Prove that \( X \) is normal.

(b) Let \( C_1 \supset C_2 \supset \ldots \supset C_n \supset C_{n+1} \supset \ldots \) be a nested sequence of closed subsets of \( X \). Prove that \( Y = \cap_{n=1}^{\infty} C_n \) is nonempty.

(c) In part (b), make the additional assumption that each \( C_n \) is connected and then prove that \( Y \) is also connected.

3. Let \( \sim \) be an equivalence relation on the compact Hausdorff space \( X \). Let \( p : X \to X/\sim \) denote the quotient map to the set of equivalence classes \( X/\sim \) equipped with the quotient topology. Recall that a subset \( B \) of \( X \) is saturated is \( B = p^{-1}(p(B)) \).

Assume that \( p \) is a closed map.

(a) Suppose that \( U \) is an open set in \( X \) containing a saturated set \( A \) of \( X \). Prove that there exists a saturated open set \( V \) of \( X \) such that \( A \subseteq V \subseteq U \).

(b) If \( X \) is compact and Hausdorff, prove that \( X/\sim \) is Hausdorff.

4. Recall that a space \( X \) is said to be first countable if at each \( x \in X \) there is some countable local base. This means, given \( x \), there is a countable collection \( \mathcal{O}_x \) of open sets containing \( x \) such that whenever \( U \) is an open set containing \( x \) there exists some \( V \in \mathcal{O}_x \) such that \( V \subseteq U \).

Let \( x \) be a point and \( A \) a subset of a first-countable space \( X \). Prove that \( x \in \overline{A} \) if and only if there exists some sequence of points in \( A \) converging to \( x \) in \( X \).