1. (a) Let \( f : X \rightarrow Y \) be a continuous map. If \( C \) is a compact subset of \( X \), prove that \( f(C) \) is compact.

(b) If \( C \) is a compact subset of \( X \) and \( X \) is Hausdorff, prove that \( C \) is a closed subset of \( X \).

2. Let \( A \) be a subset of the topological space \( X \) and define
\[
B(A) = \{ x \in X \mid U \cap A \neq \emptyset \neq U \cap (X - A) \text{ for all open neighborhoods } U \text{ of } x \}. 
\]
Show that \( A \) is both open and closed in \( X \) if and only if \( B(A) = \emptyset \).

3. Let \( f : X \rightarrow Y \) be a continuous map and let \( G = \{ (x, y) \mid y = f(x) \text{ and } x \in X \} \subset X \times Y \). Prove that \( G \) is homeomorphic to \( X \).

4. Let \( T \) be the collection of sets \( U \subset \mathbb{R}^2 \) such that \( U \) is either the empty set or for each \( (x, y) \in U \), there is an open line segment in each direction about \( (x, y) \) that is contained in \( U \).

a) Show \( T \) is a topology on \( \mathbb{R}^2 \).

b) Compare \( T \) with the standard topology; that is, is it finer, coarser, the same or none of these?

c) Let \( L \) denote a straight line in \( \mathbb{R}^2 \). Compare the subspace topology on \( L \) induced by \( T \) with the subspace topology on \( L \) induced by the standard topology on \( \mathbb{R}^2 \).

d) Let \( S \) denote a circle in \( \mathbb{R}^2 \). Compare the subspace topology on \( S \) induced by \( T \) with the subspace topology on \( S \) induced by the standard topology on \( \mathbb{R}^2 \).

5. Let \( \{X_\alpha\}_{\alpha \in I} \) be an indexed family of connected spaces and let \( X = \prod_{\alpha \in I} X_\alpha \) be the product space. Prove that \( X \) is connected.

6. Let \( Y \) denote \( S^1 \times S^1 \subset \mathbb{R}^2 \) with the subspace topology. Let \( X \) denote \( S^1 \times S^1 \) with the quotient topology induced by \( p \) where \( p : I \times I \rightarrow S^1 \times S^1 \) is defined by \( p(x, y) = (\cos(2\pi x), \sin(2\pi y)) \) and \( I = [0, 1] \subset \mathbb{R} \). Prove that \( X \) is homeomorphic to \( Y \).