Complex Functions Prelim, January 2011

Below D denotes the disk $D = \{z \in \mathbb{C} : |z| < 1\}$.
In all cases the word "analytic" is used interchangeably with "holomorphic".

1. (a) State and prove Schwarz lemma.

   (b) Let f be analytic on D with the property that $f(0) = 1$ and the real part of f is positive on D. Prove that

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$

2. In each of the cases below, determine whether there exists an analytic 1-1 mapping from U onto the complex plane. If there is, write down an explicit formula for such a mapping. Otherwise, prove that no such mapping exists.

   (a) $U = D$

   (b) $U = \{z : |z - 2| < 2\} \setminus \{z : |z - 1| \leq 1\}$.

3. Compute the following integral. Give *full justification* for your reasoning.

$$\int_0^\infty \frac{x^2 + 1}{x^4 + 1} dx.$$

4. Suppose that f, φ are analytic in a domain containing D, and that f has no zeros on ∂D. State and prove a formula for the following integral using the zeros of f.

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'}{f}(z)\varphi(z) dz.$$

5. Let f be a non constant analytic function on a domain containing 0. Assume $f(0) = 0$. Prove that for any $\delta > 0$ there exists $\epsilon > 0$ such that $f(\delta D) \supset \epsilon D$.

6. Let $\mathcal{F}$ be the family of all functions f analytic in D such that

$$\iint_D |f(x - iy)|^2 dx dy < 1.$$

   Prove that $\mathcal{F}$ is a normal family.