Below $D$ denotes the disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$.
In all cases the word “analytic” is used interchangeably with “holomorphic”.

1. (a) State and prove Schwarz lemma.
   (b) Let $f$ be analytic on $D$ with the property that $f(0) = 1$ and the real part of $f$ is positive on $D$. Prove that
   \[
   |f(z)| \leq \frac{1 + |z|}{1 - |z|}.
   \]

2. In each of the cases below, determine whether there exists an analytic 1-1 mapping from $U$ onto the complex plane. If there is, write down an explicit formula for such a mapping. Otherwise, prove that no such mapping exists.
   (a) $U = D$
   (b) $U = \{ z : |z - 2| < 2 \} \setminus \{ z : |z - 1| \leq 1 \}$.

3. Compute the following integral. Give full justification for your reasoning.
   \[
   \int_0^\infty \frac{x^2 + 1}{x^4 + 1} \, dx.
   \]

4. Suppose that $f, \varphi$ are analytic in a domain containing $D$, and that $f$ has no zeros on $\partial D$. State and prove a formula for the following integral using the zeros of $f$.
   \[
   \frac{1}{2\pi i} \int_{\partial D} \frac{f'}{f(z)} \varphi(z) \, dz.
   \]

5. Let $f$ be a non constant analytic function on a domain containing 0. Assume $f(0) = 0$. Prove that for any $\delta > 0$ there exists $\epsilon > 0$ such that $f(\delta D) \supset \epsilon D$.

6. Let $\mathcal{F}$ be the family of all functions $f$ analytic in $D$ such that
   \[
   \iint_D |f(x - iy)|^2 \, dx \, dy < 1.
   \]
   Prove that $\mathcal{F}$ is a normal family.