Some complex analysis prelim questions

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1. Suppose $f$ is a nonconstant entire function such that $f \circ f(z) = f(z)$ for all $z$. Prove that $f$ must be the identity function.

2. Suppose $f$ is entire, $f(0) = 0$ and
   \[ |f(z)| \leq e^{1/|z|} \]
   for all $z \neq 0$. Prove that $f$ is identically 0.

3. Suppose for each $n$ that $f_n$ is a bounded continuous real-valued function on the unit circle $\{z : |z| = 1\}$. Suppose for each $n$ that $u_n$ is a function that is continuous on the closed unit disk $\{z : |z| \leq 1\}$, is harmonic in the open unit disk $\{z : |z| < 1\}$, and agrees with $f_n$ on the unit circle. Show that $\{f_n\}$ is an equicontinuous family on the unit circle if and only if $\{u_n\}$ is an equicontinuous family on the closed unit disk.

4. Use residues to evaluate the definite integral
   \[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^2} \, dx. \]

5. Let $D = \{z = x + iy : 0 < y < 1, x > 0\}$. Find a conformal mapping of $D$ onto the open unit disk.

6. Suppose that for each $n$ the function $f_n$ is analytic in the open unit disk, $|f_n(0)| \leq 1$, and for each $r < 1$ satisfies
   \[ \int_{|z|=r} |f_n(z)|^2 \, |dz| \leq 1. \]
Show that every subsequence of \( \{f_n\} \) has a further subsequence which converges to a finite analytic function uniformly on each compact subset of the open unit disk.

7. Suppose for each \( n \) the function \( f_n \) is analytic on the open unit disk \( D \) and has exactly one zero in \( D \). Suppose the sequence \( \{f_n\} \) converges to \( f \) uniformly on each compact subset of the unit disk.

(a) Show that either \( f \) is identically zero on \( D \) or else has at most one zero in \( D \).

(b) Give an example of a sequence \( \{f_n\} \) where the limit function has no zeros in \( D \).