Math 5120
Preliminary Exam in Complex Analysis
August 2010

1. Characterize all those analytic functions defined in the unit disc Δ with the property that, for all $a, b \in \Delta$, $f(ab) = f(a)f(b)$.

2. Prove that there does not exist a 1-1 analytic function mapping an annulus onto a punctured disc.

3. Evaluate $\displaystyle\int_{|z|=1} \frac{z^{11}}{12z^{12} - 4z^9 + 2z^6 - 4z^3 + 1}\, dz$ and justify all steps. Hint: one of the ways to approach this problem is to make the change of variable $w = \frac{1}{z}$.

4. Suppose the sequence $\{f_n\}$ of 1-1 analytic functions converges uniformly on compact subsets of a region Ω to a function f. Show that f is analytic, and is either constant or is also 1-1.

5. Let Ω be a bounded, simply connected domain in the plane. Suppose $g : \Omega \to \Omega$ is holomorphic and not the identity. Show that g can have at most one fixed point.
(a) First show it when Ω is the unit disc. Then
(b) Show it when Ω is a bounded, simply connected region in the plane.

6. Evaluate the integral $\displaystyle\int_0^\infty \frac{\log x}{x^2 + a^2}\, dx$ where a is real and positive.

7. Prove that if f is a non-constant entire function then $f(\mathbb{C})$ is dense in $\mathbb{C}$. (You cannot just quote Picard's Theorem.)