Complex Analysis Prelim

August 23, 2007

Thursday, 9:00am – 12:00pm, MSB 118

Show all your work. Notes and textbooks are not allowed.

1. (a) Find a conformal map from the set $S = \{z : Im\, z > 0, Re\, z > 0\}$ onto the open unit disk D such that $1+\boldsymbol{i}$ is mapped into 0.
   (b) Find all the maps that satisfy (a), and prove that there are no others.

2. State and prove the Fundamental Theorem of Algebra.

3. How many zeros of the polynomial $2z^4 - z^3 + 5z^2 - 10z + 1$ lie in the set $\{z : 1 < |z| < 3\}$? (Justify your answer.)

4. Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^3 + x}\, dx$ using the Residue Theorem (justify why it can be used).
[Hint: $\frac{\sin x}{x}, x \in \mathbb{R}$, is the imaginary part of an analytic function which is bounded in the upper half-plane.]

5. Prove that if f is analytic in the open unit disk D then

$$|f(w) - f(0) - wf'(0)| \leqslant |w|^2 \sup_{z \in D} |f(z) - f(0) - zf'(0)|$$

for all $w \in D$.

6. Find all the entire functions f that satisfy $\sup \left| \dfrac{z \log |z|}{f(z)} \right| < \infty$, where the sup is over the set $\{z : z \in \mathbb{C}, z \neq 0, f(z) \neq 0\}$, and prove that there are no others.

7. Suppose that $\mathcal{F}$ is a family of analytic functions on the open unit disk such that $f(0) = 1$ for all $f \in \mathcal{F}$. Prove that if $\{Im\, f\}_{f \in \mathcal{F}}$ is a normal family, then $\mathcal{F}$ is also a normal family.