340

COMPLEX ANALYSIS PRELIMINARY EXAMINATION

August 2004

Special Instructions:

(1) Justify your answers, and show your work in the examination booklet.

(2) Notation:

- $\mathbb{C}$ the space of complex numbers
- $B(a,R) = \{z \in \mathbb{C} : |z-a| < R\}$
- $D = B(0,1)$ the unit disk in $\mathbb{C}$
- $\mathcal{H}(G)$ the space of holomorphic (analytic) functions in the region G.

1. Let f be holomorphic on $\mathbb{C}\backslash 0$. Suppose that for any $z \neq 0$
$$|f(z)| \leqslant |\log(|z|)|.$$
Show that $f(z) \equiv 0$.

2. Let C_1 and C_2 be two Euclidean circles in the plane with C_2 lying in the interior of C_1. Let Δ be the domain bounded by these circles. Is there a conformal map of Δ bijectively onto an annulus $\{z \mid 0 < r_1 < |z| < 1\}$? If there is one, describe it; if there is none, describe why none can exist?

3. Prove or give a counterexample. Suppose f is holomorphic in D and continuous on its closure. Then f extends to a holomorphic function on $B(0,R)$ for some $R > 1$.

4. Evaluate and justify your answer.
(a)
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx.$$
(b)
$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}, \quad a > b > 0.$$

5. Let $\mathcal{F} = \{f : D \to D, f \in \mathcal{H}(D)\}$, and $L = \sup_{f\in\mathcal{F}} |f'''(0)|$.
(a) Show that L exists (as a finite number).
(b) Show that there is a function $f \in \mathcal{F}$ such that $f'''(0) = L$.

6. (a) Suppose G is a bounded region in $\mathbb{C}$, $f \in \mathcal{H}(G)$, $f \neq 0$ in G, f is continuous on the closure of G, and $|f|$ is constant on the boundary of G. Prove that f is constant on G.

(b) Can the hypothesis that $f \neq 0$ in G be dropped?

(c) Can the hypothesis that G is bounded be replaced by the assumption that the complement to G is unbounded?

END