Instructions: Do all problems. Show your work in order to receive ANY credit. The terms region and domain mean the same thing. So do the terms complex analytic and holomorphic.

Problem 1: Suppose \( f \) is holomorphic in a region \( \Omega \) that contains the closed unit disk and \( |f(z)| < 1 \) when \( |z| = 1 \). How many fixed points (solutions to \( z = f(z) \)) must \( f \) have in the open unit disk \( \Delta \).

Problem 2: Suppose \( f \) is an entire function and there are constants \( A \) and \( B \) and a positive integer \( k \) so that
\[ |f(z)| \leq A + B|z|^k \]
for all \( z \). Prove that \( f \) must be a polynomial.

Problem 3: Compute (justifying your computations)

\[ (i) \quad \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + x^2} \, dx. \]

\[ (ii) \quad \int_{0}^{2\pi} \frac{d\theta}{a + b \sin \theta} \quad \text{where} \ a > b > 0. \]

Problem 4: Suppose \( f \) is holomorphic and non-zero in the simply connected domain \( \Omega \).

(i) If \( n \) is any positive integer, prove that there exists a function \( g \), holomorphic in \( \Omega \) and satisfying \( g^n = f \).

(ii) How many holomorphic solutions does \( g^3 = f \) have in a small disk about 0 if \( f(z) := z^4 + 16 \).

(iii) Find the Taylor polynomial of degree 5 for the holomorphic solution \( g \) in part (ii) for which \( g(0) \in \mathbb{R} \).

Problem 5: Suppose \( D \) is a region in \( \mathbb{C} \) and \( H(D) \) denotes the space of functions which are holomorphic in \( D \). Let \( (f_n) \) be a locally bounded sequence in \( H(D) \) and \( f \in H(D) \). Assume
\[ A := \{ x \in D \mid \lim_{n} f_n(x) = f(x) \} \]
has a limit point in \( D \). Show that there exists a subsequence of \( (f_n) \) which converges to \( f \) uniformly on compact subsets of \( D \).

Problem 6: In a domain \( D \) containing 0, a function
\[ f : \quad D \rightarrow \mathbb{C} \quad \text{with} \quad (x,y) \mapsto f(x,y) = u(x,y) + iv(x,y) \]
is complex harmonic if both \( u \) and \( v \) are (real) harmonic in \( D \). You may assume that \( f \) admits an absolutely convergent double power series expansion
\[ f(z, \bar{z}) = \sum_{n,m=0}^{\infty} a_{nm} z^n \bar{z}^m \]
and that the usual differentiation and integration rules for power series in one variable are valid here.

(i) Under what conditions on the coefficients \( a_{nm} \) is \( f \) holomorphic in \( D \)?

(ii) Under what conditions on the coefficients \( a_{nm} \) is \( f \) complex harmonic in \( D \)?