Preliminary Examination
Complex Analysis
August 25,2000

Instructions: Do all problems. Show your work in order to receive ANY credit. Where necessary, justify the validity of your answers and computations.

Problem 1: Compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\,dx.$$

Problem 2:

Let H denote the right halfplane, i.e. $H := \{z : \Re z \geq 0\}$. Given that $f : H \to H$ is holomorphic and $f(1) = 1$, show

a) $|f'(1)| \leq 1$, and

b) $\dfrac{|f(z)-1|}{|f(z)+1|} \leq \dfrac{|z-1|}{|z+1|}$.

Problem 3: Let f and g be entire functions with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant K so that $f(z) = Kg(z)$.

Problem 4: Determine the number of zeroes of the function $g(z) = e^{z-1} - az$ inside the unit circle $\{|z| < 1\}$ assuming $|a| > 1$.

Problem 5: Let $\mathcal{H}(D)$ be the set of functions holomorphic in a domain D and suppose that $\mathcal{F} \subset \mathcal{H}(D)$ is some normal family in D. Prove that $\mathcal{F}' := \{f' : f \in \mathcal{F}\}$ is also a normal family.

Problem 6: Suppose that f is entire and $f(z)$ is real if and only if z is real. Show that f can have at most one zero in $\mathbb{C}$.

Problem 7: Suppose that Δ is the unit disk and f is a holomorphic map of Δ into itself with $f(0) = 0$. If $f^{\circ n} := \underbrace{f \circ f \circ \cdots f}_{n}$, state the conditions under which $\lim_{n\to\infty} f^{\circ n}$ exists in all of Δ. When the limit does exist, what is it?