Preliminary Examination
Complex Analysis
August 25, 2000

Instructions: Do all problems. Show your work in order to receive ANY credit. Where necessary, justify the validity of your answers and computations.

Problem 1: Compute
\[ \int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 1)^2} \, dx. \]

Problem 2:
Let \( H \) denote the right halfplane, i.e. \( H := \{ z : \Re z \geq 0 \} \). Given that \( f : H \to H \) is holomorphic and \( f(1) = 1 \), show

a) \( |f'(1)| \leq 1 \), and

b) \[ \frac{|f(z) - 1|}{|f(z) + 1|} \leq \frac{|z - 1|}{|z + 1|}. \]

Problem 3: Let \( f \) and \( g \) be entire functions with \( |f(z)| \leq |g(z)| \) for all \( z \in \mathbb{C} \). Prove that there exists a constant \( K \) so that \( f(z) = K g(z) \).

Problem 4: Determine the number of zeroes of the function \( g(z) = e^{z-1} - az \) inside the unit circle \( \{ |z| < 1 \} \) assuming \( |a| > 1 \).

Problem 5: Let \( \mathcal{H}(D) \) be the set of functions holomorphic in a domain \( D \) and suppose that \( \mathcal{F} \subset \mathcal{H}(D) \) is some normal family in \( D \). Prove that \( \mathcal{F}' := \{ f' : f \in \mathcal{F} \} \) is also a normal family.

Problem 6: Suppose that \( f \) is entire and \( f(z) \) is real if and only if \( z \) is real. Show that \( f \) can have at most one zero in \( \mathbb{C} \).

Problem 7: Suppose that \( \Delta \) is the unit disk and \( f \) is a holomorphic map of \( \Delta \) into itself with \( f(0) = 0 \). If \( f^{[n]} := f \circ f \circ \cdots \circ f \), state the conditions under which \( \lim_{n \to \infty} f^{[n]} \) exists in all of \( \Delta \). When the limit does exist, what is it?