Applied Math Prelim Janurary 2016

1. Given normed linear space X,
   (a) Define weak convergence in a normed linear space and show that weak limit of a sequence is unique.
   (b) Show strong convergence implies weak convergence.
   (c) Given an example of a weak convergence sequence which is not strongly convergent.
2. Let operator A be defined by $Au = u'' + u$.
   (a) Find Green's function for operator A with $u'(0) = u(\pi) = 0$.
   (b) For $f \in L^2[0,\pi]$, define $(Tf)(x) = \int_0^\pi G(x,y)\, f(y)\, dy$. Show $Tf \in L^2[0,\pi]$.
   (c) Show $T : L^2[0,\pi] . \to L^2[0,\pi]$ is compact and find its norm.
3. Let $F : C^2\ [0,1] \to R$ be defined by
$$F(u) = \int_0^1 \sqrt{1 + (u')^2}\, dx.$$
   (a) Find Frechet derivative of F.
   (b) Find a function $u \in C^2[0,1]$ which minimized $F(u)$ among all u satisfying $u(0) = 0$, $u(1) = 2$.
4. Let H be a Hilbert space and $K : H \to H$ is linear compact operator. Show that the range of $I + K$ is closed.
5. Solve the equation $Y'' + 2Y' + Y = \delta + \delta'$ in the distributional sense, using functions of the form $Y(x) = H(x) f(x)$ where $H(x)$ is the Heaviside function.