(1) State and prove an existence theorem for the equation \( \frac{dx}{dt} + f(x) = 0 \) with initial conditions \( x(0) = 0 \) and \( x'(0) = 0 \) under the assumption that \( f \) is continuous and \( |f| \leq r \). (You can assume Rothe’s fixed point theorem.)

(2a) Find the Green’s function \( G(x,y) \) for the operator \( A \) where
\[
Au = u'' - u
\]
with \( u'(0) = u(1) = 0 \).

If \( Au = f(x) \), express the function \( u \) in terms of \( G \) and \( f \).

(2b) Define \( T : L^2(0,1) \to L^2(0,1) \) such that for any \( f \in L^2(0,1) \),
\[
(Tf)(x) = \int_0^1 G(x,y)f(y) \, dy .
\]

Explain what the spectral theorem is and why it is applicable.

(2c) Show that \( \|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } T\} \).

(2d) Compute \( \|T\| \). (hint: find eigenvalues of \( A \)).

(3) Let
\[
U(x,y) = \ln(x^2 + y^2)
\]

Compute distributionally \( \Delta U = (\partial^2_x + \partial^2_y)U \) in \( \mathbb{R}^2 \).

(4) Let \( H \) be a Hilbert space and \( K : H \to H \) is a linear, bounded, compact operator. Define \( A = I + K \). Show that if \( A \) is surjective, then it is injective.

(5) Let \( J : H^1_0(\Omega) \to \mathbb{R} \) be defined by
\[
J(u) = \int_\Omega (|\nabla u|^2/2 + u^4/4 - hu) \, dv
\]
for $h$ fixed in $L^2(\Omega)$ where $\Omega$ is a bounded region in $\mathbb{R}^3$.

(a) Find the Frechet derivative of $J$.

(b) Show that $\inf J$ is attained. You may use the fact that $H^1_0(\Omega)$ is compactly embedded in $L^t(\Omega)$ for $t < 6$ and that $\int_\Omega |\nabla u|^2 dV$ is a norm on $H^1_0(\Omega)$