Choose 5 out of the 6 questions.

(1a) State and prove an existence and uniqueness theorem for the equation \( \frac{d^2x}{dt^2} + f(x) = 0 \) with initial conditions \( x(0) = a \) and \( x'(0) = b \) under the assumption that \( f \) and its partial derivatives are continuous. (You can assume the Contraction Mapping Theorem).

(1b) Let \( a = b = f(0) = 0 \) in part (a). Can \( x(t) = t^3 \) be a solution to part (a)? Explain.

(2a) Find the Green’s function \( G(x, y) \) for the operator \( A \) where

\[
Au = -u'' + u
\]

with \( u'(0) = u'(1) = 0 \).

(2b) Define \( T : L^2(0,1) \to L^2(0,1) \) such that for any \( f \in L^2(0,1) \),

\[
(Tf)(x) = \int_0^1 G(x, y)f(y) \, dy .
\]

Explain what spectral theorem is and why it is applicable.

(2c) Show that \( \|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } T\} \).

(2d) Compute \( \|T\| \). (hint: find eigenvalues of \( A \)).
(3) Let
\[ U(x, y) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 1, \ y \geq 0, \\
0, & \text{otherwise},
\end{cases} \]
Compute its distributive derivative \( D_{xy}U \) in \( \mathbb{R}^2 \).

(4) Let \( H \) be a Hilbert space and \( K : H \to H \) is a linear, bounded, compact operator. Define \( A = I + K \). Show that if \( A \) is injective, then it is surjective.

(5) Let \( H \) be a Hilbert space and \( A : H \to H \) is compact. Show that
(a) \( x_n \rightharpoonup x \) weakly implies \( Ax_n \to Ax \).
(b) The operator norm of \( A \) is attained.

(6a) Let \( K \) be a closed convex set in a Hilbert space \( X \). Let \( x \in X \) and let \( y \) be the point of \( K \) closest to \( x \). Prove that \( \Re \langle x - y, v - y \rangle \leq 0 \) for all \( v \in K \), where \( \Re \) denotes the real part.
(6b) For each \( x \) in \( X \), we use \( Px \) to denote the point of \( K \) closest to \( x \). Using part (a) or otherwise, prove that
\[ \|Px - Pz\| \leq \|x - z\| . \]