Math 310 Preliminary Examination August 2006

DO FIVE OF THE SIX QUESTIONS!

Problem 1: (20 pts)
(a) State an existence and uniqueness theorem for the equations

\[ x' = f(x, y), \]
\[ y' = g(x, y), \]

with initial conditions \( x(0) = a \) and \( y(0) = b \) under the assumptions that \( f, g \), and all their partial derivatives are continuous.

(b) For the system:

\[ x' = x(1 - x - y), \]
\[ y' = y(1 - 2x - 3y), \]

with \( x(0) = y(0) = 1/10 \). Can either \( x(t) \) or \( y(t) \) become 0 at finite time? Justify your reasoning.

Problem 2: (20 pts)
(a) Let \( \tau_0 \in (0, 1) \). Find the Green's function for

\[ -y'' + y = \delta(t - \tau_0) \]
\[ y'(0) = y(1) = 0 \]

(b) Show that there exists a unique solution for

\[ -y'' + y = \lambda \tan^{-1} y + \cos x \]
\[ y'(0) = y(1) = 0 \]

if \( |\lambda| \) is sufficiently small.
Problem 3: (20 pts)
Prove that if an operator is of the form \( A = I + K \) where \( K \) is compact linear operator on a Hilbert space, then \( A \) is injective implies \( A \) is surjective.

Problem 4: (20 pts)
Find \( \Delta \ln(x^2 + y^2) \) in \( R^2 \) in terms of distributional derivatives.

Problem 5: (20 pts)
Let \( T \) be a compact operator on a Hilbert space \( \mathcal{H} \) and \( \{ \phi_n : n \in N \} \) be an orthonormal system of \( \mathcal{H} \).
(a) Show that \( \phi_n \rightharpoonup 0 \) weakly.
(b) Using (a) or otherwise, show that \( \lim_{n \to \infty} \| T\phi_n \| = 0 \).
(c) Let \( \lambda_n \) be a sequence of complex numbers. Show that the operator \( S \) defined by \( Sf = \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_n \rangle \phi_n \) is compact if and only if \( \lim_{n \to \infty} \lambda_n = 0 \).

Problem 6: (20 pts)

a) Suppose \( f \) is an operator from Banach space \( X \) to itself. Give the definition of \( f \) being Fréchet differentiable at a point \( x \in X \).
b) Let \( X = C[0, 1] \) with sup-norm. Let \( t_i \in [0, 1] \) and \( v_i \in C[0, 1] \), and define \( f(x) = \sum_{i=1}^{n} (x(t_i))^2 v_i \). Prove that \( f \) is Fréchet differentiable at all points of \( X \) and give a formula for \( f' \).