Math 310  Preliminary Examination  August, 2001

Answer all 5 questions.

Problem 1: (20 pts) Show that if \( p, q \) are continuous near zero, there exists an interval \((-\epsilon, \epsilon)\) and functions \( u_1 \) and \( u_2 \) such that every solution of
\[
y'' + p(x)y' + q(x)y = 0
\]
can be written as a combination of \( u_1 \) and \( u_2 \) on \((-\epsilon, \epsilon)\). Can the function \( x^5 \) solve this equation?

Problem 2: (20 pts)
(a) Find the Green's function \( g \) for the operator \( Ly = -y'' + y \) with the boundary conditions \( y'(0) = y'(1) = 0 \).
(b) Find the eigenvalues of the associated Green's operator:
\[
(Gf)(x) = \int_0^1 g(x, t)f(t) \, dt
\]

Problem 3: (20 pts)
Find the adjoint boundary value problem if
\[
Lu = u'''' + u'' + u
\]
with boundary conditions \( u(0) = u'(0) = 0, u''(1) = 0, \) and \( u(1) = 0 \).

Problem 4: (20 pts)
Let \( E \) and \( F \) be two Hilbert spaces and \( T : E \to F \) be a bounded linear operator. If there exists a complete orthonormal sequence \( \{\phi_n(x), n = 1, ..., \infty\} \) in \( E \) such that
\[
\sum_{n=1}^{\infty} \|T\phi_n\|^2 < \infty,
\]
then show that \( T \) is compact.

Problem 5: (20 pts)
Let \( U(x, y) \) be equal to 1 if \( x \geq 0 \) and \( y \geq 0 \), and zero otherwise. Find \( \frac{\partial^2 U}{\partial x \partial y} \) as a distribution.