Name: ______________________________
Section: ______________________________

Math 310 Preliminary Examination August , 2000

Problem 1: (20 pts)

(a) State and prove an existence and uniqueness theorem for the equations

$$\begin{aligned} x' &= f(x,y)\,, \\ y' &= g(x,y)\,, \end{aligned}$$

with initial conditions $x(0) = a$ and $y(0) = b$ under the assumptions that $f$, $g$, and all their partial derivatives are continuous.

(b) For the system:

$$\begin{aligned} x' &= x(1-x-y)\,, \\ y' &= y(1-2x-3y)\,, \end{aligned}$$

with $x(0) = y(0) = 1/10$. Can either $x(t)$ or $y(t)$ become 0 at finite time? Justify your reasoning.

Problem 2: (20 pts)

(a) Let $\tau_0 \in (0,1)$. Find the Green's function for

$$\begin{aligned} -y'' + y &= \delta(t-\tau_0) \\ y(0) = y'(1) &= 0 \end{aligned}$$

(b) Show that there exists a unique solution for

$$\begin{aligned} -y'' + y &= \lambda \tan^{-1} y + \cos x \\ y(0) = y'(1) &= 0 \end{aligned}$$

if $|\lambda|$ is sufficiently small.

## Problem 3: (20 pts)

Find the adjoint boundary value problem if

$$Lu = u''' + u'' + u$$

with boundary conditions $u(0) - u'(0) = 0$, $u''(0) = 0$, and $u(1) = 0$.

## Problem 4: (20 pts)

Define $M : L^2(0,1) \to L^2(0,1)$ such that for all $f \in L^2(0,1)$,

$$(Mf)(t) = (1 + \sin t)\, f(t)$$

Show that $M$ is not a compact operator.

## Problem 5: (20 pts)

Let $T$ be a compact operator on a Hilbert space $\mathcal{H}$ and $\{\phi_n : n \in N\}$ be an orthonormal system of $\mathcal{H}$.

(a) Show that $T\phi_n \rightharpoonup 0$ weakly.

(b) Using (a) or otherwise, show that $\lim_{n\to\infty} \|T\phi_n\| = 0$.

(c) Let $\lambda_n$ be a sequence of complex numbers. Show that the operator $S$ defined by $Sf = \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_n \rangle \phi_n$ is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$.