Study Guide for Ph.D. Examination in Algebra (Math 5210)

Questions on the algebra prelim exam will be drawn from the topics below. **Note**: Math 5210 will not necessarily cover every topic listed, so some amount of self-study may be required, depending on your background.

Group Theory: Homomorphisms, subgroup, quotient group, direct product. Isomorphisms and automorphisms. Group actions (esp. conjugation and left multiplication), orbit-stabilizer formula, applications to p-groups and theorems of Cauchy and Sylow. Semidirect products. Nilpotent and solvable groups. Characters of finite abelian groups and duality. Universal mapping properties of quotient group, cyclic groups, direct product, direct sum.

Examples: $\mathbf{Z}/m\mathbf{Z}$, $(\mathbf{Z}/m\mathbf{Z})^\times$, μ_m, D_n, A_n, S_n, $\mathrm{GL}_n(\mathbf{R})$, $\mathrm{SL}_n(\mathbf{R})$, $\mathrm{Aff}(\mathbf{R})$, $\mathrm{Heis}(\mathbf{R})$, and similar finite matrix groups over $\mathbf{Z}/m\mathbf{Z}$ (especially $\mathbf{Z}/p\mathbf{Z}$).

Ring Theory: Characteristic, integral domain, field, direct product. Homomorphisms, subring, ideals and quotient rings, prime ideals and maximal ideals, nilpotent elements, Zorn's lemma. Chinese remainder theorem. PID, UFD. Universal mapping properties of $\mathbf{Z}$, quotient ring, direct product ring, and fraction field of a domain.

Examples: $\mathbf{Z}$, $\mathbf{Z}/m\mathbf{Z}$, $F[X]$, $F[X]/(f)$, $\mathbf{Z}[i]$ (and other $\mathbf{Z}[\sqrt{d}]$), $F(X)$, $\mathbf{Z}[X]$, $F[X,Y]$, $F[X,Y]/I$.

Modules: Free modules, cyclic modules, direct sums. Homomorphism, submodule, quotient module. Vector spaces: basis, dimension, linear transformations (including matrix representation using a basis), characteristic polynomial, eigenvalues and eigenvectors. Universal mapping properties of free module, quotient module, direct sum, direct product.

Examples: $\mathbf{Z}$-modules (abelian groups), F-modules (vector spaces), $F[X]$-modules, ideals (principal and non-principal).

References:

D. Dummit and R. Foote, *Abstract Algebra*, 3rd ed., Wiley, 2003.

T. Hungerford, *Algebra*, Springer-Verlag, 1980.

S. Lang, *Algebra*, 3rd revised ed., Springer-Verlag, 2005.

J. Rotman, *Advanced Modern Algebra*, Prentice Hall, 2002.