Justify all your steps. You may use any results that you know unless the question says otherwise, but don’t invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. Let $G$ be a group and $Z$ be its center.
   (a) Prove that if $G/Z$ is cyclic then $G$ is abelian.
   (b) Prove that if $G$ is a nontrivial finite $p$-group then $Z$ is nontrivial.
   (c) Prove that if $G$ is a nontrivial finite $p$-group and every abelian normal subgroup of $G$ is contained in $Z$ then $G$ is abelian. (Hint: If $G$ were not abelian, look at the center of $G/Z$. Ideas from parts a and b may be useful here.)

2. The prime factorization of 2015 is $5 \cdot 13 \cdot 31$. Prove every group of order 2015 has a cyclic normal subgroup of order $13 \cdot 31 = 403$, and explain why a non-abelian group of order 2015 exists.

3. Let $R$ be a commutative ring with unity, and let $I$ be a proper ideal of $R$.
   (a) Prove there is a one-to-one correspondence between the ideals of the quotient ring $R/I$ and the ideals of $R$ that contain $I$.
   (b) Under the correspondence in part a, prove prime ideals of $R/I$ correspond to prime ideals of $R$ that contain $I$.

4. Let $R$ be an integral domain.
   (a) Let $a, b \in R$ and assume $a$ is a unit. Prove the substitution homomorphism $\varphi: R[x] \to R[x]$ where $\varphi(f(x)) = f(ax + b)$ for all $f(x) \in R[x]$ is an automorphism of $R[x]$ and it fixes each element of $R$.
   (b) Prove a converse to part a: every ring automorphism of $R[x]$ that fixes each element of $R$ is of the form described in part a.

5. Let $R$ be a commutative ring with identity. An element $e \in R$ is called an idempotent if $e^2 = e$. Prove that if $I$ and $J$ are ideals in $R$ such that $R = I \oplus J$ (that is, $R = I + J$ and $I \cap J = \{0\}$) then there is an idempotent $e$ in $R$ such that $I = Re$ and $J = R(1 - e)$. (Hint: If $e$ is an idempotent, then $1 - e$ is also an idempotent.)

6. Give examples as requested, with brief justification.
   (a) An automorphism of the group $(\mathbb{Z}/11\mathbb{Z})^\times$ other than the identity and inversion.
   (b) A prime factorization of 15 in $\mathbb{Z}[i]$.
   (c) An explicit construction of a field of size 9.
   (d) An irreducible cubic polynomial in $\mathbb{F}_5[x]$. 